Inverse Matrix, Norms, and Condition Numbers

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files: ConditionNumber.mw, ConditionNumber.pdf
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First load the LinearAlgebra library, and create a square matrix **A**:

> $with(LinearAlgebra):$

> $A := Matrix([[0.835, 0.667], [0.333, 0.266]])$

$$A := \begin{bmatrix} 0.835 & 0.667 \\ 0.333 & 0.266 \end{bmatrix} \tag{1}$$

Calculate the determinant of **A** and its inverse, which we'll call **AI**. We'll confirm the inverse by multiplying **A** and **AI**.

> $Determinant(A)$
> \# Notice that the determinant is very small. This suggests that matrix **A** is ill-conditioned.

$$-0.000001 \tag{2}$$

> $AI := MatrixInverse(A)$ \# This is $\mathbf{A}^{-1}$, except for possible truncation error.

$$AI := \begin{bmatrix} -2.65999999993532\ 10^{5} & 6.66999999983782\ 10^{5} \\ 3.32999999991903\ 10^{5} & -8.34999999979697\ 10^{5} \end{bmatrix} \tag{3}$$

> $Multiply(A, AI)$ \# This should produce the 2 x 2 identity matrix **I**, except for possible truncation error.

$$\begin{bmatrix} 0.999999999970896 & 1.16415321826935\ 10^{-10} \\ 0. & 1. \end{bmatrix} \tag{4}$$

Define a constant vector **b1**, and use LinearSolve to solve the system **Ax** = **b1**.
We'll call the solution **x1**:

> $b1 := Vector([0.168, 0.067])$

$$b1 := \begin{bmatrix} 0.168 \\ 0.067 \end{bmatrix} \tag{5}$$

> $x1 := LinearSolve(A, b1)$ \# This is the solution of system **Ax** = **b1**.
> It is the vector $[1, -1]$ except for trunction error.

$$x1 := \begin{bmatrix} 0.999999999972231 \\ -0.999999999965236 \end{bmatrix} \tag{6}$$

To confirm that **x1** is the solution, we'll multiply **A** and **x1**. We should get the constant vector **b1** except for truncation error.

> $Multiply(A, x1)$

$$\begin{bmatrix} 0.168000000000000 \\ 0.0670000000000001 \end{bmatrix} \quad (7)$$

As further confirmation, we'll multiply $\mathbf{A}^{-1}$ and **b1**. We should get the solution vector **x1** except for truncation error.

> $soln := Multiply(AI, b1)$

$$soln := \begin{bmatrix} 0.999999999978172 \\ -0.999999999978172 \end{bmatrix} \quad (8)$$

Let's now define a slightly different constant vector **b2** and solve the system **Ax = b2**. We'll call the solution **x2**:

> $b2 := Vector([0.167, 0.067])$

$$b2 := \begin{bmatrix} 0.167 \\ 0.067 \end{bmatrix} \quad (9)$$

> $x2 := LinearSolve(A, b2)$ # This is the solution of system **Ax = b2**.
It is the vector [267, -334] except for trunction error.

$$x2 := \begin{bmatrix} 266.999999993511 \\ -333.999999991877 \end{bmatrix} \quad (10)$$

To confirm that **x2** is the solution, we'll multiply **A** and **x2**. We should get the constant vector **b2** except for truncation error.

> $Multiply(A, x2)$

$$\begin{bmatrix} 0.167000000000030 \\ 0.0670000000000073 \end{bmatrix} \quad (11)$$

As further confirmation, we'll multiply $\mathbf{A}^{-1}$ and **b2**. We should get the solution vector **x2** except for truncation error.

> $soln := Multiply(AI, b2)$

$$soln := \begin{bmatrix} 266.999999993510 \\ -333.999999991880 \end{bmatrix} \quad (12)$$

In the preceding examples, we solved systems **Ax = b1** and **Ax = b2** where the input vectors **b1** and **b2** differ by only one digit in the first component. Compare **b1** and **b2** again:

> $b1;\ b2$

$$\begin{bmatrix} 0.168 \\ 0.067 \end{bmatrix}$$

$$\begin{bmatrix} 0.167 \\ 0.067 \end{bmatrix} \quad \textbf{(13)}$$

That very small change to the input vector produced a VERY LARGE change in the solution **x**. Compare solutions **x1** and **x2** again:

> *x1*; *x2*

$$\begin{bmatrix} 0.999999999972231 \\ -0.999999999965236 \end{bmatrix}$$

$$\begin{bmatrix} 266.999999993511 \\ -333.999999991877 \end{bmatrix} \quad \textbf{(14)}$$

Why did a very small change to the input vector produce a very large change in the solution? Because the coefficient matrix **A** is ill-conditioned. Recall that a matrix is ill-conditioned if its condition number is much larger than 1. To see this, we'll calculate the condition number of **A** using the infinity norm.

> *ConditionNumber*(*A*, *infinity*) # Note that the condition number of **A** is much larger than 1.

$$1.754336000\ 10^{6} \quad \textbf{(15)}$$

Recall that the condition number of **A** is defined by $\|\mathbf{A}\| \cdot \|\mathbf{A}^{-1}\|$. Let's verify this:

> *ANorm* := *Norm*(*A*, *infinity*) # This is the infinity norm of **A**.

$$ANorm := 1.502 \quad \textbf{(16)}$$

> *AINorm* := *Norm*(*AI*, *infinity*) # This is the infinity norm of $\mathbf{A}^{-1}$.

$$AINorm := 1.16799999997160025\ 10^{6} \quad \textbf{(17)}$$

> *ANorm* · *AINorm* # This should equal the condition number of **A** that we obtained above.

$$1.754336000\ 10^{6} \quad \textbf{(18)}$$

Let's now calculate the condition number of **A** using the Frobenius norm.

> *ConditionNumber*(*A*, *Frobenius*) # Note that the condition number of **A** is much larger than 1.

$$1.323759000\ 10^{6} \quad \textbf{(19)}$$

> *ANorm* := *Norm*(*A*, *Frobenius*) # This is the Frobenius norm of **A**.

$$ANorm := 1.150547261 \quad \textbf{(20)}$$

> *AINorm* := *Norm*(*AI*, *Frobenius*) # This is the Frobenius norm of $\mathbf{A}^{-1}$.

$$AINorm := 1.15054726106128586\ 10^{6} \quad \textbf{(21)}$$

> *ANorm* · *AINorm* # This should equal the condition number of **A** that we obtained above.

$$1.323759000\ 10^{6} \quad \textbf{(22)}$$
