Kettering University
14 th Mathematics Olympiad

November 22, 2014

Problems and Solutions

Problem 1. Solve the equation $x^2 - x - \cos y + 1.25 = 0$.

Solution.

$$\begin{aligned} x^2 - x - \cos y + 1.25 &= x^2 - x + 0.25 + 1 - \cos y \\ &= (x - 0.5)^2 + (1 - \cos y) \end{aligned}$$

$$(x - 0.5)^2 + (1 - \cos y) = 0$$

Since $1 - \cos y \geq 0$,

$$\begin{cases} x - 0.5 = 0 \\ 1 - \cos y = 0 \end{cases}$$

$$\begin{cases} x = 0.5 \\ \cos y = 1 \end{cases}$$

$$\textbf{Answer}: \begin{cases} x = 0.5 \\ y = 2\pi n, n \text{ is an arbitrary integer.} \end{cases}$$

Problem 2. Solve the inequality:

$$\left| \frac{x-2}{x-3} \right| \leq x$$

Solution.

$$\begin{cases} x \geq 0 \\ -x \leq \frac{x-2}{x-3} \leq x \end{cases}$$

The first case: $0 \leq x < 3$.

$$-x(x-3) \geq x - 2 \geq x(x-3)$$

$$\begin{cases} x^2 - 2x - 2 \le 0 \\ x^2 - 4x + 2 \le 0 \end{cases}$$

a. $x^2 - 2x - 2 \le 0$.

$$x^2 - 2x - 2 = 0$$

$$x = 1 \pm \sqrt{3}$$

$$1 - \sqrt{3} \le x \le 1 + \sqrt{3}$$

b. $x^2 - 4x + 2 \le 0$.

$$x^2 - 4x + 2 = 0$$

$$x = 2 \pm \sqrt{2}$$

$$2 - \sqrt{2} \le x \le 2 + \sqrt{2}$$

Combining these two cases one gets:

$$2 - \sqrt{2} \le x \le 1 + \sqrt{3}$$

This interval is inside the interval $[0, 3)$.
The second case: $x > 3$.

$$-x(x-3) \le x - 2 \le x(x-3)$$

$$\begin{cases} x^2 - 2x - 2 \ge 0 \\ x^2 - 4x + 2 \ge 0 \end{cases}$$

a. $x^2 - 2x - 2 \ge 0$.

$$x \le 1 - \sqrt{3} \text{ or } x \ge 1 + \sqrt{3}$$

b. $x^2 - 4x + 2 \ge 0$.

$$x \le 2 - \sqrt{2} \text{ or } x \ge 2 + \sqrt{2}$$

Combining these two cases one gets:

$$x \le 1 - \sqrt{3} \text{ or } x \ge 2 + \sqrt{2}$$

Since $x > 3$,

$$x \geq 2 + \sqrt{2}$$

Answer: $[2 - \sqrt{2}, 1 + \sqrt{3}] \cup [2 + \sqrt{2}, +\infty)$.

Problem 3. Bilbo and Dwalin are seated at a round table of radius R. Bilbo places a coin of radius r at the center of the table, then Dwalin places a second coin as near to the table's center as possible without overlapping the first coin. The process continues with additional coins being placed as near as possible to the center of the table and in contact with as many coins as possible without overlap. The person who places the last coin entirely on the table (no overhang) wins the game.

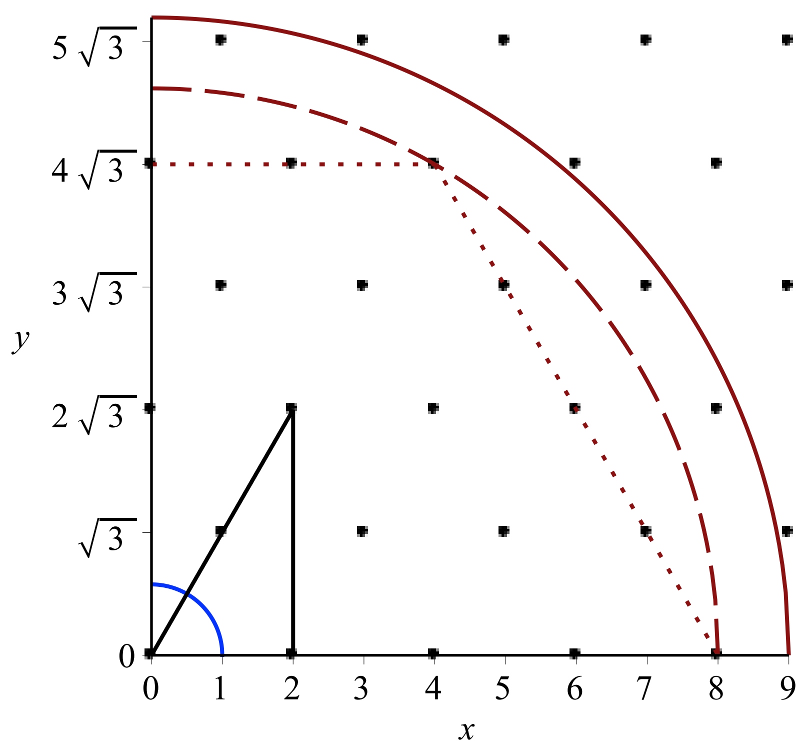
Assume that R/r is an integer.

(a) Who wins, Bilbo or Dawalin? Please justify your answer.

(b) How many coins are on the table when the game ends?

Solution to #3 on the Olympiad

Consider the figure of a quarter circle with a hexagonal lattice.



The above diagram illustrates 1/4 of a circular table of radius R = 9. The black dots represent the centers of coins. We assume without loss of generality that the coins have radius 1. The blue circle show 1/4 of a coin. The dashed 1/4 circle of radius 8 is the center boundary which we denote as Rc - thus a coin is completely on the table (no overlap) iff its center is within or on this boundary. Our goal is to determines how many centers are within or on the dashed boundary. The dotted line shows the face of a hexagon which we will discuss later.

<u>Part(a)</u>

Symmetry implies that excluding the coin at the origin, the table must contain an even number of coins. Hence the total number of coins on the table must be odd, thus the first player wins the game.

<u>Part(b)</u>

We first consider centers with y-values of $2 \cdot j \cdot \sqrt{3}$, $j = 0, 1, 2, ..$ A line from the origin to this type of center forms a 30 - 60 right triangle with base and height of $2\,k$, **and** $2 \cdot j \cdot \sqrt{3}$ respectively for k = 1, 2, 3,.. and j = 1, 2, 3.. The square of the hypotenuse is $4 \cdot \left(k^2 + 3 \cdot j^2\right)$, hence this center is on or within the dashed boundary iff $4 \cdot \left(k^2 + 3 \cdot j^2\right) \le Rc^2$ or since k must be an integer we have

$$k \le \left[\sqrt{\frac{Rc^2}{4} - 3 \cdot j^2} \right]$$

where [z] = the largest integer less than or equal to z. The total number of centers on "even levels", thus $y = 2 \cdot j \cdot \sqrt{3}$, $j = 0, 1, 2, ..$ is obtained by summing over all such levels, thus

$$Ne = \sum_{j=0}^{Je} \left(\left[\sqrt{\frac{Rc^2}{4} - 3 \cdot j^2} \right] + 1 \right)$$

where $Je = \left[\frac{Rc}{2 \cdot \sqrt{3}} \right]$ and *Ne* equals the total number of centers on "even levels" within or on the dashed boundary. The " + 1" in the above results counts the centers on the y-axis. A way of obtaining the value of *Je* is to note that it must be an integer and the argument of the square root can not be less than zero. The above expression can be written as

$$Ne = 1 + Je + \frac{Rc}{2} + \sum_{j=1}^{Je} \left(\left[\sqrt{\frac{Rc^2}{4} - 3 \cdot j^2} \right] \right)$$

A similar analysis of centers with y-values of $(2 \cdot j - 1) \cdot \sqrt{3}$, $j = 1, 2, 3, ,$ - *odd levels,* shows that the number of such centers within or on the dashed boundary is given by

$$No = \sum_{j=1}^{Jo} \left[\frac{\left(\sqrt{Rc^2 - 3 \cdot (2 \cdot j - 1)^2} + 1 \right)}{2} \right]$$

where $Jo = \left[\frac{Rc}{2 \cdot \sqrt{3}} + \frac{1}{2} \right]$. Note that *Jo* is the largest integer for which the argument of the square root is non-negative.

Adding the above two results gives the total number of centers within or on Rc on a quarter circle. Thus

$$Nq = 1 + Je + \frac{Rc}{2} + \sum_{j=1}^{Je} \left(\left[\sqrt{\frac{Rc^2}{4} - 3 \cdot j^2} \right] \right) + \sum_{j=1}^{Jo} \left(\left[\frac{\left(\sqrt{Rc^2 - 3 \cdot (2 \cdot j - 1)^2} + 1 \right)}{2} \right] \right)$$

Comment: The above result applied to Rc = 8 gives the correct value of 19 as shown in the above figure.

To obtain the number of centers (coins) on an entire circle we note that in the above result "1" represents the center at the origin; *Je and Rc* represent the number of centers (excluding the center at the origin) on the y-axis and x-axis respectively hence both must be multiplied by a factor of 2; and the sums gives the number of centers in the 1st quadrant excluding the aforementioned centers thus must be multiple by a factor of 4. This gives the total number of centers within or on the Rc boundary, or equivalently the number of coins which completely fit on a table of radius R = Rc + 1 as

$$N = 1 + 2\cdot Je + Rc + 4\left(\sum_{j=1}^{Je}\left(\left\lfloor\sqrt{\frac{Rc^2}{4} - 3\cdot j^2}\right\rfloor\right) + \sum_{j=1}^{Jo}\left(\left\lfloor\frac{\left(\sqrt{Rc^2 - 3\cdot(2\cdot j - 1)^2} + 1\right)}{2}\right\rfloor\right)\right) \tag{1}$$

Some example

A table of radius equal to 3: Rc = 2 and the total number of coins that completely fit on this table is 7.

A table of radius equal to 9: Rc = 8 and the total number of coins that completely fit on this table is 61.

A table of radius equal to 15: Rc = 14 and the total number of coins that completely fit on this table is 187.

The above result is general - it holds for any value of Rc. However it is somewhat complicated. In some cases the answer takes on a simpler form. For example if Rc = 2, it is obvious that 6 coins form a circle around the central coin giving a total of 7 coins. In addition, as shown in the above diagram, the number of coins that fit on the table equals the number of centers contained within the hexagon inscribed within the circle of radius Rc. This equality will hold as long as there exist no centers between the face of the hexagon and the Rc-boundary.

The following question is relevant. How many coins are contained within a hexagon?

Consider a hexagon composed of horizontal layers with L-coins per side. The top layer contains L coins, the next contains L+1 coins, ...the central layer contains L + ([L-1]). Hence the top half of the hexagon, excluding the central layer contains L + (L+1) + (L+2) + ... + (L +[L-2]) = L(L-1) +(1 + 2+ ... + [L-2]). We know that

$$\sum_{k=1}^{M} k = \frac{M(M+1)}{2}$$

Hence the sum (1 + 2+ ... + [L-2]) = $\frac{(L-2)\cdot(L-1)}{2}$. This give the total number of coins in the hexagon excluding the central layer as $2\cdot\left(L\cdot(L-1) + \frac{(L-2)\cdot(L-1)}{2}\right) = 3L^2 - 5L + 2$

The central layer contains L+(L-1), hence the total number of coins in a hexagon with L coins per side is $3L^2 - 3L + 1$. It should be clear from the above diagram that the distance from center of the hexagon to it corner is $2\cdot L - 2$ (units of coin radius r = 1) which equals the center boundary radius Rc. Thus $L = \frac{Rc}{2} + 1$. Substituting this into the above result for the total number of coins (centers) contained in a hexagon of side L gives,

$$Nh = \frac{3}{4}Rc^2 + \frac{3}{2}Rc + 1$$

(2)

Eq(2) gives the number of coins contained within a hexagon with "center distance" Rc and equals the number of coins (centers) on a table with center boundary Rc ***as long as no centers exist between the the face of the hexagon and the Rc boundary.*** For example if Rc = 8, Eq(2) gives 61 which agrees with the results of Eq(1).

To this point we have two equations, Eq(1) which is complicated, but gives the correct answer for all values of Rc. Eq(2) which has a simple form, however we expect it will yield incorrect results when Rc becomes too large, thus when centers inhabit the region between the hexagonal face and the Rc boundary.

Question: when does Eq(2) break-down?

The face of the hexagon intersects the kth - level, $y = k\cdot\sqrt{3}$, at an x-value $Rc - k,\ 0 \le k \le \left\lfloor \frac{Rc}{\sqrt{3}} \right\rfloor$ which is obvious from the above figure. The Rc boundary intersects the kth level at an x-value of $\sqrt{Rc^2 - 3\cdot k^2}$. The distance from the hexagonal face boundary to the Rc boundary is obtained by subtracting these two quantities. The coin centers are separated by a distance of 2. Hence a center exists in this region when

$$\sqrt{Rc^2 - 3\cdot k^2} - (Rc - k) \le 2 .$$

Simplifying this equation gives

$$k^2 \le \frac{(k-2)\cdot(2\cdot Rc - (k-2))}{3} \quad \textbf{for } 2 < k \le \left\lfloor \frac{Rc}{\sqrt{3}} \right\rfloor \tag{3}$$

.

where we note that the rhs must be greater than zero. For Rc = 10, k has possible value 3, 4, and 5, none of which satisfy the above inequality. Similarly Rc = 12 does not satisfy Eq(3). However, Eq(3) is satisfied for Rc = 14 with k-values of 3, 4, and 5. Thus Rc = 14 is the smallest value of Rc for which Eq.(2) does not give the correct answer. In fact for Rc = 14 Eq.(1) gives N = 187, while Eq.(2) gives Nh = 169.

Result

For all values of Rc (recall Rc must be an even integer), Eq(1) gives the correct number of coins that completely fit on a table of radius Rc + 1.

For $2 \le Rc \le 12$, Rc even, Eq(2) gives the correct number of coins that completely fit on a table of radius Rc + 1.

Approximate result which becomes more accurate as R increases.

In the above figure, connecting each center with a straight line produces a set of parallelograms each with area of $2 \cdot \sqrt{3}$. Each center is a corner of four separate parallelogram, and since there are four corners one center is associated with one parallelogram. Hence an approximation to the number of centers contained within the Rc boundary is given by the area enclosed by the Rc boundary divided by $2 \cdot \sqrt{3}$,

$$N \approx \frac{\pi \cdot Rc^2}{2 \cdot \sqrt{3}} \tag{4}$$

The above result only approximates the number of centers within or on the Rc boundary, N, because some of the centers close to the boundary will be associated with parallelograms which are only partially within the boundary. The number of *problem centers* will decrease as a percentage of the total number of centers contained within the Rc boundary as Rc increases (why?), thus the above approximation becomes more accurate as Rc increases.

Some examples

Rc = 14 , Eq(1) gives N = 187, Eq(4) approximates N as 177.75

Rc = 28, Eq(1) gives N = 721, Eq(4) approximates N as 711

Rc = 56, Eq(1) gives N = 2857, Eq(4) approximates N as 2844

Finished! And all this because Bilbo challeged Dwalin to a *rigged game of coins on a round table.* Perhase next year Dwalin will challeng Bilbo to a game of coins on a triangular table, or to a game of checker, chess, or backgammon. Who knows? See you next year.

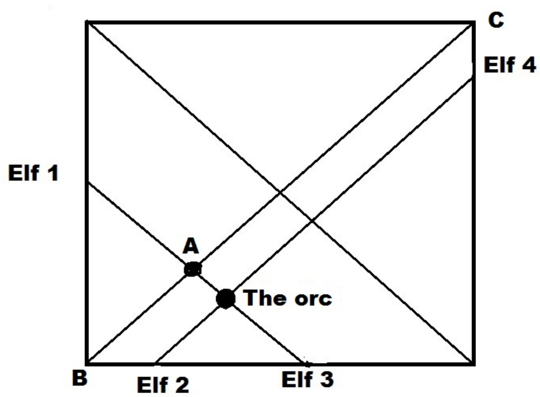


Figure 2: Problem 4.

Problem 4. In the center of a square field is an orc. Four elf guards are on the vertices of that square. The orc can run in the field, the elves only along the sides of the square. Elves run 1.5 times faster than the orc. The orc can kill one elf but cannot fight two of them at the same time. Prove that elves can keep the orc from escaping from the field.

Solution.

Consider the lines passing through the position of the orc and parallel to the diagonals of the square. Elves should take positions at the points of intersection of these lines with the sides of the square. Let A be a projection of the position of the orc on the diagonal BC. To keep his position the Elf 3 has to have the same projection on BC. To achieve this he has to move $\sqrt{2}$ times faster that the orc. It is possible since the elf can move 1.5 times faster than the orc. Similarly for the other elves.

Problem 5. Nine straight roads cross the Mirkwood which is shaped like a square, with an area of 120 square miles. Each road intersects two oposite sides of the square and divides the Mirkwood into two quadrilateras of areas 40 and 80 square miles. Prove that there exists a point in the Mirkwood which is an intersection of at least three roads.

Solution.

Let us consider a road KL that intersects sides AD and BC. It divides the square on two trapesoids $DKLC$ and $AKLB$ the areas of which are related

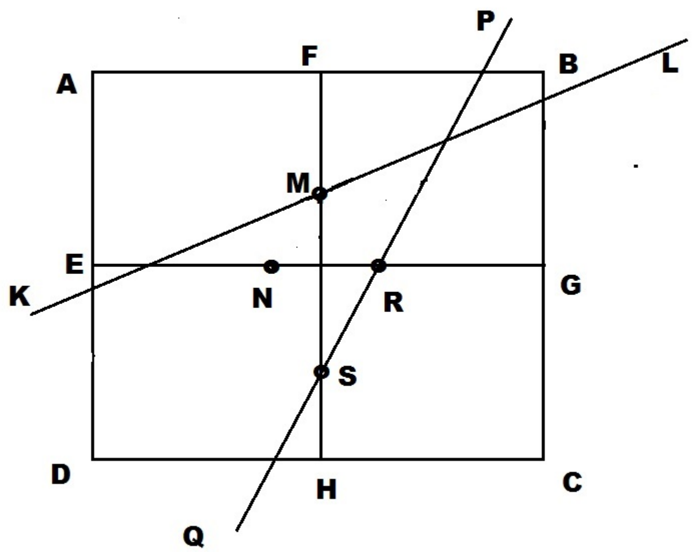


Figure 3: Problem 5.

as 1:2. Therefore the middle lines of triapesoids FM and MH are related as 1:2. Consider points M, S, N, and R that divides FH and EG in three equal segment. Then each road should pass through one of this four points. Since there are nine roads through one of them at least three roads should pass.

Problem 6. Thirteen dwarfs visited Bilbo's house. In front of each of them Bilbo put a big mug. Then he opened 4 gallon barrel of wine and poured it all into the mugs. When Bilbo left the room Thorin divided the content of his mug into 12 equal parts and poured it into the mugs of the other dwarfs so that each one of them got 1/12th of Thorin's wine. After he finished, the second dwarf, Balin, divided the content of his mug into 12 equal parts and poured it into the mugs of the other dwarfs so that each of them got 1/12th of the Balin's wine. Then the third dwarf did the same, then the fourth and so on. After all dwarfs completed this action they discovered to their great surprise that everybody has the same amout of wine he had before Bilbo left the room. How was wine finally distributed among the dwarfs?

Solution.

Denote by y_m gal. the amount of wine the dwarf number m had originally, when Bilbo left the rom.

Denote by x gal. the maximal amount of wine that was in any mug during all these pourings of wine.

Assume that it was in the mug of a dwarf number m.

Since there were $m-1$ dwarfs before him and each dwarf had no more than x gal. in his mag,

$$x - y_m \leq \frac{x(m-1)}{12}.$$

$$y_m \geq x - \frac{x(m-1)}{12} = \frac{x(13-m)}{12}.$$

When this dwarf divided his wine there was zero amount of wine in his mag. Then remaining $13-m$ dwarfs poured wine in his mug.

Each of them had less than or equal to x gal. of wine, so the amouunt of wine mth dwarf got was less than or equal to $x(13-m)/12$ gal.

Since finally he got y_m gal. of wine,

$$y_m \leq \frac{x(13-m)}{12}.$$

Therefore,

$$y_m = \frac{x(13-m)}{12}.$$

So each dwarf poured $x/12$ gal. in his mag, which means that each dwarf had x gal. at least once.

Therefore, the previous formula is true for every $m = 1, 2, \ldots, 13$.

Thus,

$$x + \frac{11x}{12} + \frac{10x}{12} + \ldots + \frac{x}{12} + 0 = 4.$$

Therefore,

$$\frac{x(1+2+\ldots+12)}{12} = 4.$$

Since,

$$1+2+\ldots+12 = 12 \cdot \frac{1+12}{2}$$

$$\frac{13x}{2} = 4.$$

Thus, $x = 8/13$, and

$$y_m = \frac{2(13-m)}{39}.$$