Kettering University Mathematics Olympiad For High School Students 2008

1. (*Solution by Neil Gurram, a 4th-7th finisher*)
   We claim that Mr. Brown is the criminal.

   First, we show that Mr. Brown lied one time and told the truth once, Mr. Potter told a lie twice and Mr. Smith told the truth both times.

   - We show that this situation is possible by assuming that Smith told the truth twice. Then, Brown committed the crime and Smith did not.
   - This truth would imply that Mr. Potters is lying for both statements, as he claimed that Mr. Brown did not commit the crime and Mr. Smith did, which means he is lying.
   - Hence, Mr. Brown is telling a lie and a truth. From the previous statements, one sees that he is lying about "his not doing it" and telling the truth that Mr. Potter did not do it. So, Mr. Brown lied once and told the truth once.

   Therefore from these statements, Mr. Brown committed the crime.

   Now, we must show that the other two did not commit the crime. Therefore, either Mr. Brown or Mr. Potter told the truth twice.

   - If Mr. Brown told truth twice "I have not done it, Mr. Potter has not done it". Then Mr. Potter has to telling the truth twice as then Mr. Smith must have committed the crime and Mr. Brown did not do it, so Mr. Potter told the truth twice, a contradiction.
   - If Mr. Potter told truth twice "Mr. Brown has not done it. Mr. Smith has done it." Then since Mr. Smith committed the crime, then neither Mr. Brown nor Mr. Smith committed the crime so Mr. Brown is telling the truth twice, a contradiction.

   Therefore, Mr. Brown is the culprit.

2. (*Solution by Randy Jia, a 4th-7th finisher*)
   We will prove that, in general, with an odd number of figures, it is not possible for every figure to touch exactly 3 other figures. If this is proved, the problem is trivial.

We will make this problem a graph theory problem.

So we have $2n+1$ vertices, each which represents n figure. (a circle in this case) We connect an edge between 2 vertices if the 2 figures touch each other.

Definition: The degree of a vertex is the number of edges coming out of that vertex or the number of vertices it is connected to.

In our problem, our $2n+1$ vertices each have degree 3. So
Total Degree $= \sum \deg = (2n+1) \times 3 = 6n+3 =$odd.

However, an edge is the connection of 2 vertices, so half of the total degrees is the number of edges.

But $\frac{6n+3}{2} = \frac{\text{odd number}}{2}$, so not an integer, which is impossible. Thus, it is impossible to have an odd number (1000001) of circles such that each circle touches exactly 3 other circles.

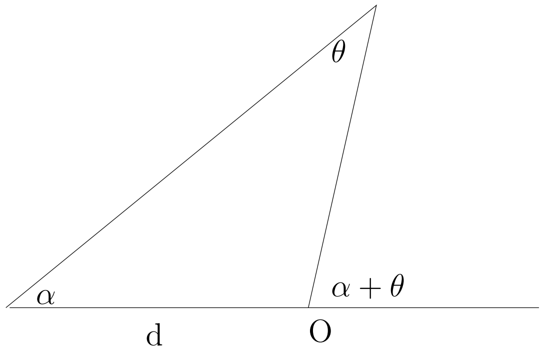
3. (*Solution by Nicholas Triantafillou, a 4th-7th finisher*)
We note that since the angle of incidence stays the same with each reflection, the eight points must be spaced evenly around the circle. Denote these points (clockwise) by $P_1, P_2, \ldots P_8$. We note that to hit all 8 points, the path must cycle in of the following patterns since the number of points skipped between consecutive points visited is constant.
Patterns: a)$P_1P_4P_7P_2P_5P_8P_3P_6$, b)$P_1P_2P_3P_4P_5P_6P_7P_8$

These patterns cycle around and can also be taken in the opposite order.

In pattern a, the angle of incidence is $\frac{1}{2} \times \frac{2\pi}{8} = \frac{\pi}{8}$ radians.

In pattern b, the angle of incidence is $\frac{1}{2} \times \frac{6\pi}{8} = \frac{3\pi}{8}$ radians.

We note that if the angle of incidence is θ and the particle is launched at angle α from point $(-\alpha, 0)$ then the angle between the x-axis and the point at which the particle is reflected is $\theta + \alpha$, since the exterior angle equals the sum of the opposite interior angles. Now, we will find ,given angle α and point $(-\alpha, 0)$, what radius allows us to have angle of incidence θ. From above, we know that the intersection of the lines $y = \tan\alpha(x+d)$ and $y = \tan(\alpha+\theta)x$ lies on the circle.



Their intersection point can be easily shown to be

$$x_0 = \frac{d\tan\alpha}{\tan(\alpha+\theta)-\tan(\alpha)},\ y_0 = \frac{\tan\alpha d\tan(\alpha+\theta)}{\tan(\alpha+\theta)-\tan(\alpha)}.$$

Then $R^2 = x_0^2 + y_0^2$ since the origin is $(0,0)$, so

$$\begin{aligned} R &= \frac{d\tan\alpha}{\tan(\alpha+\theta)-\tan(\alpha)}\sqrt{1+\tan^2(\alpha+\theta)} \\ &= \pm\frac{d\tan\alpha\sec(\alpha+\theta)}{\tan(\alpha+\theta)-\tan(\alpha)} \\ &= \frac{\pm d\sin\alpha}{\cos\alpha\sin(\alpha+\theta)-\sin\alpha\cos(\alpha+\theta)} \\ &= \frac{\pm d\sin\alpha}{\sin\theta}. \end{aligned}$$

Thus $\sin alpha = \pm\frac{R}{d}\sin\theta$. So $\alpha = \pm\sin^{-1}\left(\pm\frac{R}{d}\sin\theta\right)$. Since θ can equal $\frac{\pi}{8}$ or $\frac{3\pi}{8}$, $\alpha = \pm\sin^{-1}\left(\pm\frac{R}{d}\sin\frac{\pi}{8}\right)$ or $\alpha = \pm\sin^{-1}\left(\pm\frac{R}{d}\sin\frac{3\pi}{8}\right)$. When both are defined, both solutions work, when only one is defined, only that solution works and when neither is defined, there are no solutions.

4. (*Solution by Andrew Jeanguenat, Andrew placed third in the 2008 Kettering Math Olympiad*)
Since w, x, y, z are distinct and the 4 equations are symmetric, we can assume WLOG that $w > x > y > z$. Also, since $z^3 = w^2 + x^2 + y^2$, and $w^2 + x^2 + y^2 \geq 0, z \geq 0$, if $z = 0$ then $0 = x^2 + y^2 + z^2 \Rightarrow x, y, z = 0$. This is a contradiction, therefore $z > 0$.

Thus $w > x > y > z > 0$.

Now, assume that $w \geq 3$ then $w^3 = x^2+y^2+z^2 < 3w^2$. Thus $w^3 < 3w^2$ or $w^2(w-3) < 0$. However, if $w \geq 3$, $w^2(w-3) \geq 0$. This is a contradiction. Therefore, $0 < w < 3$.

Assume $w < 3$. Then we know $w^3 = x^2 + y^2 + z^2$, $x^3 = w^2 + y^2 + z^2$, $y^3 = w^2 + x^2 + z^2$, and $z^3 = w^2 + x^2 + y^2$. So
$w^3 + x^3 + y^3 + z^3 = 3w^2 + 2x^2 + 3y^2 + 3z^2$
$\Rightarrow (w^3 - 3w^2) + (x^3 - 3x^2) + (y^3 - 3y^2) + (z^3 - 3z^2) = 0$.

However, if $0 < a < 3$ then $a^2(a-3) < 0 \Rightarrow a^3 - 3a^2 < 0$. Thus the 4 terms in parentheses above are all less than 0. This is a contradiction. Therefore, it is impossible to find 4 distinct real numbers such that the cube of every number equals the sum of squares of the other 3 numbers.

5. (*Solution by David Sherman, David came in 2nd in the 2008 Kettering Math Olympiad*)
We first prove a lemma:
Lemma: Given $n \in N$, there is a Fibonacci number f_k such that $\frac{1}{2}n < f_k \leq n$.
Proof: First note that for $n = 1$, $f_1 = 1$ suffices. When $n = 2, f_2 = 2$ suffice. Thus assume statement true for $n > 2$. Consider all Fibonacci numbers less than or equal to $n : 1, 2, f_3, \ldots f_k$. Suppose BWOC that $f_k \leq \frac{1}{2}n$. Then consider $f_{k-1} + f_k = f_{k+1}$. Since $f_{k-1} \leq f_k \leq \frac{1}{2}n$ we have $f_{k+1} = f_{k-1} + f_k \leq \frac{1}{2}n + \frac{1}{2}n = n$. Then our list was not complete. This is a contradiction and the lemma is proven.

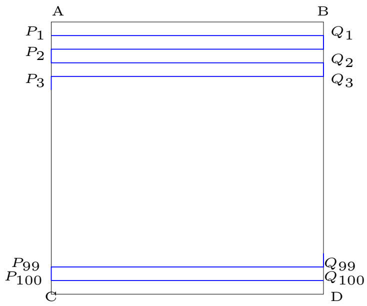
We prove the actual result by strong induction on n. As a base case, consider $n = 1 = f_1$.

Now, assume the claim for all cases up to some $n - 1$. Consider case $n > 2$. By the lemma, $\exists k$ such that $\frac{1}{2}n > f_k \leq n$. Then $n - f_k$ is an integer such that $0 \leq n - f_k < \frac{1}{n}$. If $n - f_k = 0$ use $n = f_k$ and we are done. Otherwise $1 \leq n - f_k < \frac{1}{2}n \leq n - 1.(truebecausen\geq 2)$. By the induction hypothesis, $n - f_k$ is the sum of unique Fibonacci numbers: $n - f_k = f_\alpha + f_\beta + \cdots + f_\omega$, $1 \leq f_\alpha < f_\beta < \cdots < f_\omega$. Then obviously $n = f_\alpha + f_\beta + \cdots + f_\omega + f_k$. We need only check that it is distinct.

Suppose BWOC that two of the Fibonacci numbers in the sum are equal. Then obviously f_k is one of the two since the rest are given to be distinct. We would have $f_i = f_k$ where $i \in \{\alpha, \beta, , \omega\}$. Then since f_i

are all positive and $n - f_k = f_\alpha + f_\beta + \cdots + f_\omega$ we must have $f_i < n - f_k$ but $n - f_k < \frac{1}{2}n$ and $\frac{1}{2}n < f_k$ thus $f_i < n - fr_k < \frac{1}{2}n < f_k \Rightarrow f_i < f_k$. This is a contradiction and so it is must be distinct.

6. (*Solution by Allen Yuan, Allen is the winner of the 2008 Kettering Math Olympiad*) There always exists such a broken line.
Proof: (all number have units of m) Square ABCD is the unit square.



Consider the broken line $Q_1P_1P_2A_2Q_3P_3P_4 \ldots P_{100}Q_{100}$ which is outlined in blue such that $AP_1 = BQ_1 = DQ_{100} = CP_{100} = \frac{1}{200}$ and $P_1P_2 = P_2P_3 = \ldots = P_{99}P_{100} = Q_1Q_2 = \ldots Q_{99}Q_{100} = \frac{1}{100}$. All P_iQ_i are parallel to AB and CD. The length of this broken line is $101 - \frac{1}{100} < 101$.

Now, note that any point in $ABCD$ is within $\frac{1}{200}$ of this broken line. Therefore, start at Q_1 and walk on the path. Every time time you are within $\frac{1}{200}$ of a point (directly above or below) walk to the point and then an arbitrarily small distance to the side, then walk directly back to the line. The extra length you travel is at most $2 \times \frac{1}{200} = \frac{1}{100}$ per point. Since there are 10000 points, the total extra length you travel is at most $10000 \times \frac{1}{100} = 100$ and because all points in $ABCD$ are within $\frac{1}{200}$ of the line, so we will hit all points. The total path, then is at most $\left(101 - \frac{1}{100}\right) + 100 < 201$. Hence, this path that you walked is the desired broken line.