Kettering University Mathematics Olympiad For High School Students 2006, Sample Solutions

1. (*Solution by Roger Jia, a 4th-7th finisher*)
Let $2x$ miles be the distance from one city to the other. Thus, for the first half of the distance, x miles, the speed is 600mi/hr. The time, in hours, is time$=\frac{\text{distance}}{\text{speed}} = \frac{x}{600}$hrs.

   For the second half, we find time$=\frac{\text{distance}}{\text{speed}} = \frac{x}{900}$hrs.

   Hence, average speed time$=\frac{\text{total distance}}{\text{total time}} = \frac{2x}{\frac{x}{600}+\frac{x}{900}} = \frac{2x}{\frac{5x}{1800}} = 720$mi/hr.

2. (*Solution by Andrew Jeanguenat, a 4th-7th finisher*)

   (a) If there were to be a decrease in Carton A, one egg from Carton A must be selected and one empty space from Carton B must be selected. If ordered of selection mattered, you could either pick one of the 3 eggs and then one of the 9 blank spaces in Carton B or you could pick one of the blank spaces in Carton B and then 1 of the 3 eggs in Carton A., Therefore, there should be 3(9)+9(3)=54 possible selections if ordered matters, but since order doesn't matter each selection is being double counted. Therefore, there are 27 possible selections/interchanges.

   (b) Let the number of eggs in Carton A$=x$. Then, the number of eggs in carton B$=6-x$. So the number of spaces in Carton A$=6-x$. Number os spaces in Carton B$=12-(6-x)=6+x$. For there to be an increase in A, 1 empty space from A and 1 egg from B must be chosen. This results in $(6-x)(6-x)$ selections. For there to be an increase in Carton B, 1 egg from A must be chosen and 1 empty space from B must be chosen. This results in $x(6+x)$ selections. So $(6-x)(6-x)=x(6+x) \Rightarrow x^2-12x+36=x^2+6x \Rightarrow 18x=36 \Rightarrow x=2$. Number of increases for $A=(6-2)(6-2)=16$. Number of increases for $B=2(6+2)=16$.

   Therefore, 2 eggs should be placed in Carton A, and 4 eggs should be placed in Carton B.

3. (*Solution by Ram Bhaskar, a 4th-7th finisher*)
We can break up each square with side a into 4 squares with side $\frac{a}{2}$ (as shown in diagram). The subdivision is labeled in the diagram.

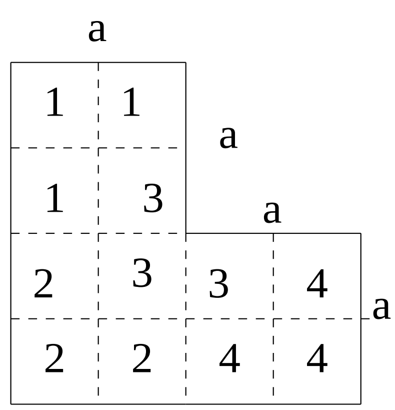

Figure 1: Solution for Problem 3

4. (*Solution by Philip Hu, Philip placed third in the 2007 Kettering Math Olympiad.*)
   Let $x = \sqrt[3]{5\sqrt{2}+7} - \sqrt[3]{5\sqrt{2}-7}$. Then,

$$\begin{aligned} x^3 &= 5\sqrt{2}+7-3(5\sqrt{2}+7)^{\frac{2}{3}}(5\sqrt{2}-7)^{\frac{1}{3}}+3(5\sqrt{2}+7)^{\frac{1}{3}}(5\sqrt{2}-7)^{\frac{2}{3}}-5\sqrt{2}+7 \\ &= 14-3(5\sqrt{2}+7)^{\frac{1}{3}}(5\sqrt{2}-7)^{\frac{1}{3}}\Big(\underbrace{(5\sqrt{2}+7)^{\frac{1}{3}}-(5\sqrt{2}-7)^{\frac{1}{3}}}_{x}\Big) \\ &= 14-3\Big((5\sqrt{2})^2-7^2\Big)x \end{aligned}$$

   where the last step substitutes in the value of x. Since $(5\sqrt{2})^2 - 7^2 = 1$ we have $x^3 = 14 - 3x$. Note that factoring $x^3 + 3x - 14 = 0$ gives $(x-2)(x^2 + 2x + 7) = 0$. Since $x^2 + 2x + 7 = 0$ has imaginary solutions of $-1 \pm i\sqrt{6}$, $x = 2$ is the only <u>real</u> solution. In addition since $\sqrt[3]{5\sqrt{2}+7} - \sqrt[3]{5\sqrt{2}-7}$ is clearly real, we see that the numerical value is indeed 2.

5. (*Solution by Neil Gurram, Neil placed second in the 2007 Kettering Math Olympiad*)
   Let AD, BE and CF be the medians. Furthermore, we assume $AD = 15$, $BE = 12$ and $CF = 9$. Also, being medians $CD = DB$, $BF = FA$, $AE = EC$.

   We will now, try to find a relationship between a triangle with sides equal to the length of the medians and $\triangle ABC$, which had medians AD, BE and CF.

   (a) We can translate CD to got to F' as shown in figure. Notice that $CDF'F$ is a parallelogram because CD is translated to FF'.

   (b) Also, we have $BF'FD$ as a parallelogram because $FF'||CD$ and C, D and B are collinear, so $FF'||BD$ and theat $FF' = CD = BD$.

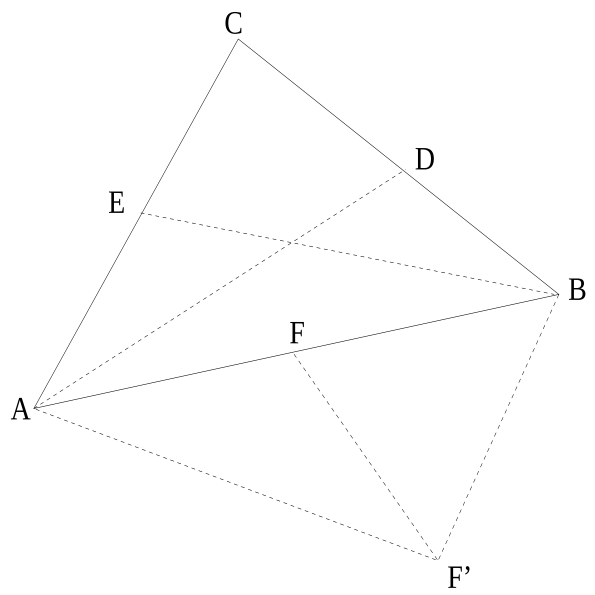

Figure 2: Solution for Problem 5

(c) Now, notice that $BF'||DF$ from (b). Also, $\frac{BD}{BC} = \frac{BF}{BA} = \frac{1}{2}$ and $\angle DBF = \angle CBA$, so $\triangle DBF \sim \triangle CBA$. Hence, $\angle DFB = \angle CAB$, so $CA||FD$. But C, E, A are collinear, so $AE||FD$ and $AE = \frac{1}{2}AC = DF$ by Mid-line Theorem, so $AE = DF = BF'$. But because $\vec{BF'}||\vec{DF}$ and $\vec{AE}||\vec{DF}$ where $\vec{AE} \neq \vec{BF'}$, then $AEBF'$ has to be a parallelogram, for opposite sides are parallel and equal. Thus $BE = AF'$.

(d) Now, notice that we have constructed $DF' = CF'$, $AD = AD$ and $BE = AF'$ from (c). So ADF' is a triangle which has sides at length equal to the medians of $\triangle ABC$.

(e) We denote $[A]$ as the area of region A.
Notice that $CDF'F$ being a parallelogram implies that $[DFF'] = [CDF] = \frac{1}{2}[BCF]$ because D is a midpoint at BC and $\frac{1}{2}[BCF] = \frac{1}{4}[ABC]$ because F is a midpoint of AB. The reason why $[ACF] = [BCF]$ immediately follows from $A = \frac{1}{2}bh$ where the height is constant in both triangles as well as the base. So $[DFF'] = \frac{1}{4}[ABC]$.

(f) Likewise, we have $[ABE] = [ABF']$ because $AF'BE$ is a parallelogram. But we know that with E being a midpoint at AC, $[ABE] = \frac{1}{2}[ABC]$ and we know that F being a midpoint at AB implies that $[AFF' = \frac{1}{2}[ABF'] = \frac{1}{2}[ABE] = \frac{1}{4}[ABC]$.

(g) Finally we have $[ADF] = \frac{1}{2}[ADB] = \frac{1}{4}[ABC]$ both on account that F is the midpoint of AB and D is the midpoint of BC. So $[ADF] = \frac{1}{4}[ABC]$.

(h) Now we notice that $\triangle ADF, \triangle DFF', \triangle AFF'$ do not overlap except on edges. So $[ADF'] = [ADF] + [DFF'] + [AFF'] = \frac{3}{4}[ABC]$.

(i) However, we know that $[ADF']$ is equal to the area of a triangle with the side length as the medians of $\triangle ABC$. So we now have to find the area of a triangle with sides at length 9,12 and 15, where $AD = 15, AF' = 12, DF' = 9$. Notice that $9^2 + 12^2 = 15^2$ so by Pythagorean Theorem, $\triangle ADF'$ is a right angle triangle with legs at length 9 and 9 and hypotenuse at length 15. So the area of $\triangle ADF'$ is $\frac{1}{2}(9)(12) = 54cm^2$.

(j) Finally $[ADF'] = 54 \Rightarrow 54 = \frac{3}{4}[ABC]$ from (h), so $[ABC] = \frac{4}{3}(54) = 72cm^2$.

Thus the area of the triangle with median of length 9,12 and 15 is $72cm^2$.

6. (*Solution by Nicholas Triantafillou, Nicholas is the winner of the 2007 Kettering Math Olympiad.*)
We will prove that any listing of the first $n^2 + 1$ integers contains some $n + 1$ integers in either increasing or decreasing order by induction. The case for $n = 9$ is requested in this problem.
Our base case is $n = 1$. Then, the lot is either 1,2 or 2,1 both which clearly contain (1+1)=2 integers in such an order.
Assume that $k + 1$ such integers exist among any $k^2 + 1$ listed.
We will proceed by contradiction. Suppose that we have a listing of $(k + 1)^2 + 1$ integers with no $k + 2$ ordered integers. We will denote the ith term in out list as a_i.
Consider the right most $k^2 + 1$ integers. i.e. a_{2k+2} to a_{k^2+2k+2}. These must contain some $k + 1$ ordered integers from our induction hypthoesis. Without loss of generality, due to similar argument, let the order be decreasing. Let the greatest (left most) term among these $k + 1$ integers be b_1. Clearly, $a_1, a_2, \ldots, a_{k+1}$ must either all be less than b_1 or we have a list of $k + 2$. Now ignore b_1 on our list, choosing the right most $k^2 + 1$ integers, with the exception of b_1. (a_{2k+1} to a_{k^2+2k+2}). Again we have $k + 1$ ordered integers. We have two cases.
Case 1: The new sequence is decreasing. Then call the leftmost term of the new sequence b_2. Since we have no sequences of length $k + 2$, b_2 is to the left of b_1 if $b_2 < b_1$ and to the right of b_1 if $b_1 < b_2$.
Case 2: The new sequence is increasing. Then call the left most term of the new sequence c_1. Then $a_1, a_2, \ldots a_{2k}$ must all be greater than c_1.

Now we ignore our renamed terms, and again repeat the process, changing subscripts appropriately. Once a term has been added to the c sequence, case 2 should be read identically to case 1 except for replacing b by c and switching the $<$ and $>$ signs with each other.

This process creates 2 lists of terms. The b list in increasing order and with the leftmost and smallest term greater than our remaining unchosen terms on the range a_1 to a_{2k+2} and the c list in decreasing order and with the right most and the largest term greater than the remaining unchosen terms on the range. After a_2 has been included, we have completed $(2k+2)-2+1 = 2k+1$ iterations, and by Pigeonhole Principle, one list contains $k+1$ numbers.

If it is the b list, since the list is increasing and all terms are greater than a_1 we have a sequence of $k+2$ and a contradiction.

Similarly, if it is the c list, since the list is decreasing and all terms are less than a_1 we have a sequence of $k+2$ so we have a contradiction in all cases. Thus, among any n^2+1 distinct integers, some $n+1$ must be ordered either positively or negatively for any integer n. For the problem, this means that there is a way to cross out 72 numbers from our list of $82 = 9^2+1$, leaving $10 = 9+1$ that are ordered.