Kettering University Mathematics Olympiad For High School Students 2006

1. At a conference a mathematician and a chemist were talking. They were amazed to find that they graduated from the same high school. One of them, the chemist, mentioned that he had three sons and asked the other to calculate the ages of his sons given the following facts:

   (a) their ages are integers,

   (b) the product of their ages is 36,

   (c) the sum of their ages is equal to the number of windows in the high school of the chemist and the mathematician.

   The mathematician considered this problem and noted that there was not enough information to obtain a unique solution. The chemist then noted that his oldest son had red hair. The mathematician then announced that he had determined the ages of the three sons. Please (aspiring mathematicians) determine the ages of the chemists three sons and explain your solution.

2. A square is inscribed in a triangle. Two vertices of this square are on the base of the triangle and two others are on the lateral sides. Prove that the length of the side of the square is greater than and less than 2r, where r is a radius of the circle inscribed in the triangle.

3. You are given any set of 100 integers in which none of the integers is divisible by 100. Prove that it is possible to select a subset of this set of 100 integers such that their sum is a multiple of 100.

4. Find all prime numbers a and b such that $a^b + b^a$ is a prime number.

5. N airports are connected by airlines. Some airports are directly connected and some are not. It is always possible to travel from one airport to another by changing planes as needed. The board of directors decided to close one of the airports. Prove that it is possible to select an airport to close so that the remaining airports remain connected.

6. (A simplified version of the Fermat's Last Theorem). Prove that there are no positive integers x, y, z and $z \leq n$ satisfying the following equation:

$$x^n + y^n = z^n.$$