Kettering University Mathematics Olympiad For High School Students 2005, Sample Solutions

1. (*Solution by Meelap Shah, a 4th-6th finisher*)
   Expanding the left side, we get

   $$1+x^2+x^4+x^6 = 4x^3 \Rightarrow (x^6-2x^3+1)+(x^4-2x^3+x^2) = 0 \Rightarrow (x^3-1)^2+x^2(x-1)^2 = 0.$$

   Since all terms on the left hand side are non-negative, they both must be zero. Hence we have

   $$\begin{aligned} & (x^3-1)^2 = 0 \Rightarrow x^3 = 1 \Rightarrow x = 1. \\ \text{and} \quad & x^2(x-1)^2 = 0 \Rightarrow x = 0 \text{ or } x = 1. \end{aligned}$$

   So the only possible solution is $x = 1$.

2. (*Solution by Frederic Sala, a 4th-6th finisher*)
   We show that since Nick goes first, there is a guaranteed strategy for him to win. Let Nick takes 1 pebble on his first turn, leaving 99. now, if John takes x pebbles ($1 \leq x \leq 8$), Nick will take $9-x$ pebbles. So for each pair of turns, 9 pebbles will be removed. After 10 turns, there will be $99 - 9(10) = 0$ pebbles left. Whatever number, between 1 and 8, of pebbles John takes, Nick can take the remaining number and win.

   John can only be guaranteed to win if Nick doesn't take 1 pebble on the first turn. If Nick takes between 2 and 8 pebbles, there will be between 92 and 98 pebbles left. John should take enough to be left with 90. Thereafter, John should take $9-x$, where x is the number of pebbles that Nick took. Again, we will be down to 9 pebbles when it is Nick's turn, guaranteeing John a win. In general, either player should try to leave a multiple of 9 pebbles for his opponent. The first to do so wins by following the strategy above.

3. (*Solution by Kevin Dilks, Mr Dilks placed third in the 2005 Kettering Math Olympiad.*)
   First, note that $cos 12x = cos^2 6x - \sin^2 6x = 1 - 2\sin^2 6x$. So we need to show that

   $$\sin x(\sin x + \sin 3x + \sin 5x + \cdots \sin 11x) = \sin^2 6x.$$

   We see that

   $$\begin{aligned} & \sin x = \sin(6x-5x) = \sin 6x \cos 5x - \sin 5x \cos 6x \\ & \sin 11x = \sin(6x+5x) = \sin 6x \cos 5x + \sin 5x \cos 6x \\ \Rightarrow \quad & \sin x + \sin 11x = 2\sin 6x \cos 5x. \end{aligned}$$

   Similar we can be obtain the following identities:

   $$\begin{aligned} \sin 3x + \sin 9x &= 2\sin 6x \cos 3x, \\ \sin 3x + \sin 9x &= 2\sin 6x \cos x. \end{aligned}$$

Hence we need to show that

$$2\sin x\sin 6x(\cos x+\cos 3x+\cos 5x)=\sin^2 6x.$$

Or equivalently, we need to show that

$$2\sin x(\cos x+\cos 3x+\cos 5x)=\sin 6x.$$

$$\begin{aligned} & \cos 5x=\cos(4x+x)=\cos 4x\cos x-\sin 4x\sin x \\ & \cos 3x=\cos(4x-x)=\cos 4x\cos x+\sin 4x\sin x \\ \Rightarrow\quad & \cos 3x+\cos 5x=2\cos 4x\cos x. \end{aligned}$$

So

$$2\sin x\cos x(1+2\cos 4x)=2\sin x\cos x[1+2(2\cos^2 2x-1)]=\sin 2x(4\cos^2 2x-1).$$

Working with the right hand side, we have

$$\begin{aligned} \sin 6x=\sin(4x+2x) &= \sin 4x\cos 2x+\sin 2x\cos 4x \\ &= 2\sin 2x\cos^2 2x+\sin 2x\cos 4x \\ &= \sin 2x(2\cos^2 2x+\cos 4x) \\ &= \sin 2x(2\cos^2 2x+2\cos^2 2x-1) \\ &= \sin 2x(4\cos^2 2x-1) \end{aligned}$$

which equals the left hand side. Thus the equality holds.

4. (*Solution by Bohao Pan, a 4th-6th finisher*)
   Nick starts off with 7 pieces. Suppose Nick cuts n_1 pieces. Then he now has:
   $7-n_1+7n_1=7+6n_1$ pieces.

   In the next turn, Nick cuts n_2 pieces. He now has:
   $(7+6n_1)-n_2+7n_2=7+6(n_1+n_2)$ pieces.

   In the *kth* turn, Nick cuts n_k pieces. He now has:
   $[(7+6(n_1+n_2+\cdots+n_{k-1})]-n_k+7n_k=7+6(n_1+n_2+\cdots n_k)$ pieces.

   Nick claims that he now has 2000 pieces. That is,

   $$7+6(n_1+n_2+\cdots n_k)=2000\Rightarrow 6(n_1+n_2+\cdots n_k)=1993.$$

   Since 1993 is not divisible by 6, no integer values $n_1,n_2,\ldots n_k$ satisfies the above. Thus John concludes that Nick made an error.

5. (*Sample Solution:*)From the shown diagram, we see that

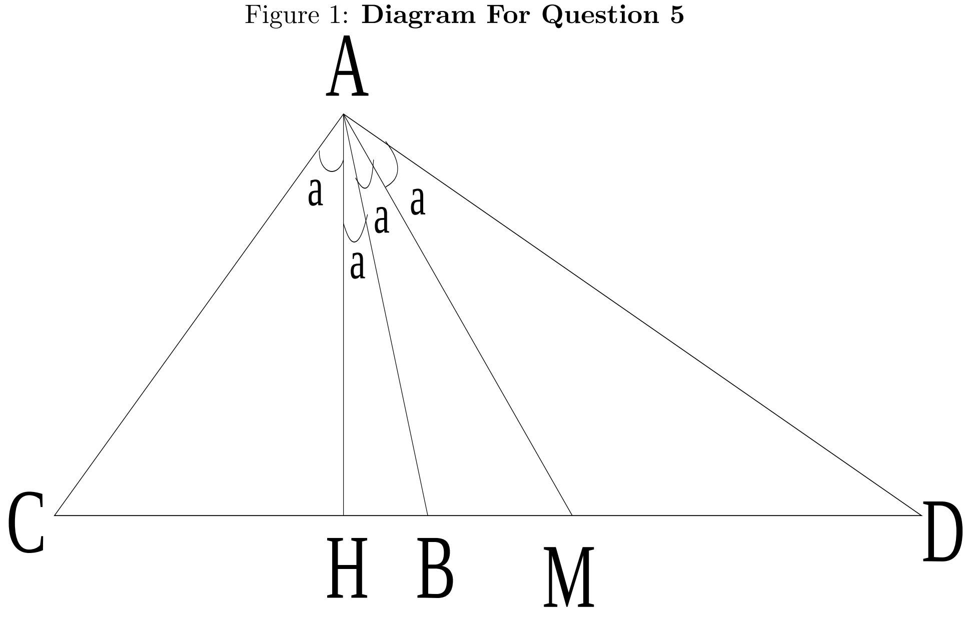
$$
\begin{aligned}
& \text{Area}(ACM) = \text{Area}(AMD) \\
\Rightarrow\quad & \frac{1}{2}|AC||AM|\sin(3a) = \frac{1}{2}|AM||AD|\sin(a) \\
\Rightarrow\quad & |AC|\sin(3a) = |AD|\sin(a) \\
\Rightarrow\quad & |AD| = \frac{\sin(3a)}{\sin(a)}|AC|
\end{aligned}
$$

Also

$$
\begin{aligned}
& |AH| = |AC|\cos(a) = |AD|\cos(3a) \\
\Rightarrow\quad & |AD| = \frac{\cos(a)}{\cos(3a)}|AC| \\
\Rightarrow\quad & \frac{\sin(3a)}{\sin(a)} = \frac{\cos(a)}{\cos(3a)} \\
\Rightarrow\quad & \sin(3a)\cos(3a) - \sin(a)\cos(a) = 0 \\
\Rightarrow\quad & \frac{1}{2}\sin(6a) - \frac{1}{2}\sin(2a) = 0 \\
\Rightarrow\quad & \frac{1}{2}\Big(\sin(2a)\cos(4a) + \sin(4a)\cos(2a) - \sin 2a\Big) = 0 \\
\Rightarrow\quad & \frac{1}{2}\Big(\sin(2a)\cos(4a) + 2\sin(2a)\cos^2(2a) - \sin 2a\Big) = 0 \\
\Rightarrow\quad & \sin(2a)\cos(4a) = 0
\end{aligned}
$$

where $0 < 4a < \pi \;\Rightarrow 0 < a < \frac{\pi}{4} \;\Rightarrow \sin(2a) \neq 0$. Hence $\cos(4a) = 0 \Rightarrow 4a = \frac{\pi}{2} \Rightarrow a = \frac{\pi}{8}$.

Figure 1: **Diagram For Question 5**



6. (a) (*Solution by Alex Xu, Mr Xu placed second in the 2005 Kettering Math Olympiad.*)

First every city must contain a connection. Since two cities require one line to be connected, adding a city adds a minimum of one line in order for the new city to be connected. Thus, for n cities, a minimum of $n-1$ connectors is needed just to connect all of the cities. For $k = 1$, each city must be connected to every other city. Thus $\begin{pmatrix} 100 \\ 2 \end{pmatrix} = \frac{(100)(99)}{2} = 4950$ airlines are needed. For $k = 2$, each city can be connected to a single city. This uses $n - 1$ connections, which is the minimum number of connections. This setup also satisfies all $k > 2$.

One possibles scheme is to connect city 1 to every other city. This gives 99 connections.

(b) (*Sample Solution:*) If all cities are connected to each other, there is nothing to prove. So let's divide the 100 cities into two distinct groups, A and B respectively, where no cities in group A is connected to cities in group B. Let the number of cities in group A be k, then the number of cities in group B is $100 - k$. The total number of airlines needed to ensure that all cities in group A is connected directly is:

$$(k-1) + (k-2) + \cdots + 1 = \frac{k(k-1)}{2}.$$

Similarly, The total number of airlines needed to ensure that all cities in group B is connected directly is:

$$(99-k) + (98-k) + \cdots + 1 = \frac{(99-k)(100-k)}{2}.$$

The total number of airlines needed thus far is:

$$T(k) = \frac{k(k-1)}{2} + \frac{(99-k)(100-k)}{2} = k^2 - 100k + 4950 = (k-50)^2 + 2450, \quad 1 \leq k \leq 99.$$

The above parabola attains its maximum in the allowable range of k at $k = 1$ and $k = 99$ with T(1)=T(99)=4851. So with 4852 airlines we can use the remaining airline to connect any city in group A to group B.