Kettering University Mathematics Olympiad For High School Students 2004, Sample Solutions

1. (*Solution by Mr. Dan Pan, a 4th-6th finisher*)
   By inspection, we see that $(x, y) = (0, 1)$ and $(x, y) = (1, 0)$ are solutions.

   The solutions of $x^6 + y^6 = 1$ are bounded by $-1 \leq 1$ and $-1 \leq y \leq 1$ because both x^6 and y^6 are nonnegative numbers. Thus, its graph is bounded similarly as well.

   $x^6 + y^6 = 1 \rightarrow y = \pm(1 - x^6)^{\frac{1}{6}}$. The corresponding graph is shown in Figure 1. $x^5 + y^5 = 1 \rightarrow y = (1 - x^5)^{\frac{1}{5}}$. The corresponding graph is

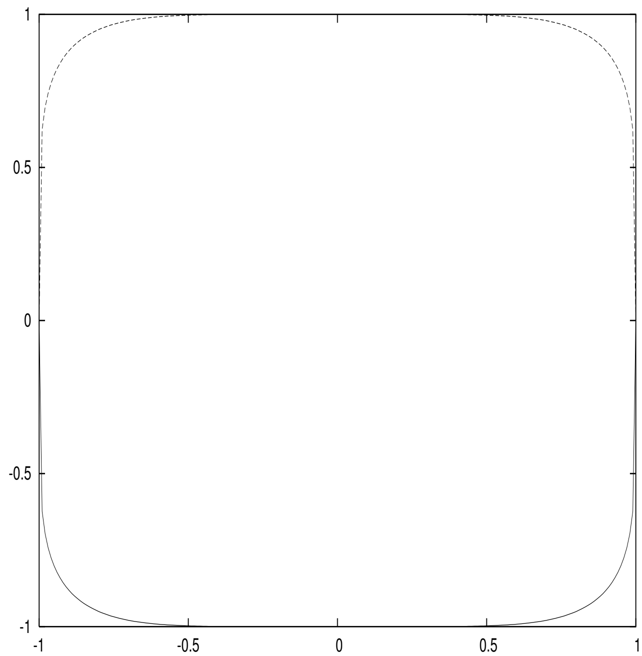

Figure 1: Graph of $x^6 + y^6 = 1$

   shown in Figure 2. From above we see that when $x < 0, y > 1$ and when $y < 0, x > 1$. These conditions mean that the solutions of the system are bounded by $0 \leq x \leq 1$ and $0 \leq y \leq 1$. (because the graph of $y = (1-x^5)^{\frac{1}{5}}$ will not intersect with $y = (1 - x^6)^{\frac{1}{6}}$ outside of these bounds)

   Now we find all solutions within the bounds:

   - $(0, 1)$, $(1, 0)$ are known solutions
   - Note 1: for $x \in (0, 1)$, $x^6 < x^5$
   - Note 2: for $x \in (0, 1)$, $1 - x^6 > 1 - x^5$
   - Note 3: for $a \in (0, 1)$, $a^{\frac{1}{6}} > a^{\frac{1}{5}}$

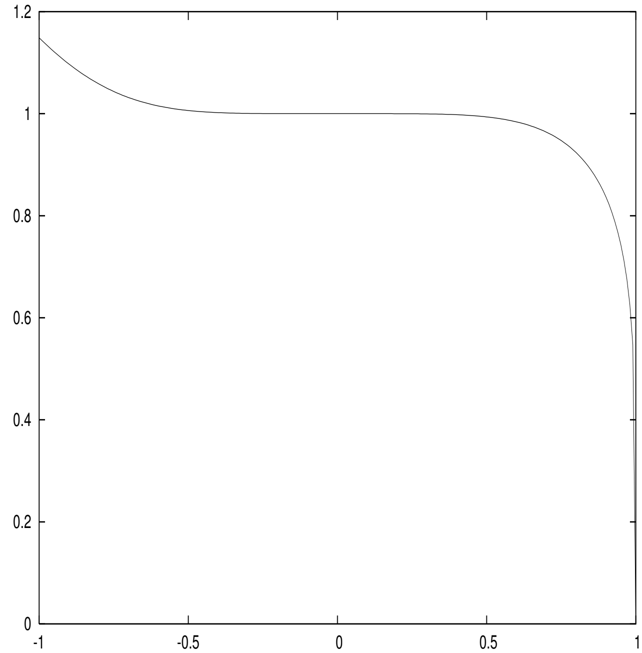

Figure 2: Graph of $x^5 + y^5 = 1$

- Note 2 and 3 implies that for $x \in (0,1)$, $(1-x^6)^{\frac{1}{6}} > (1-x^5)^{\frac{1}{5}}$
- Thus, the curve $y = (1-x^6)^{\frac{1}{6}} > y = (1-x^5)^{\frac{1}{5}}$ for all $x, y \in (0,1)$ and thus there are no intersections/solutions. Therefore, $(x,y) = (0,1)$ and $(x,y) = (1,0)$ are the only solutions of the system.

2. (*Solution by Mr. Rahul Ramesh, a 4th-6th finisher*)
   Using the diagram drawn in Figure 3 we can conclude the following:

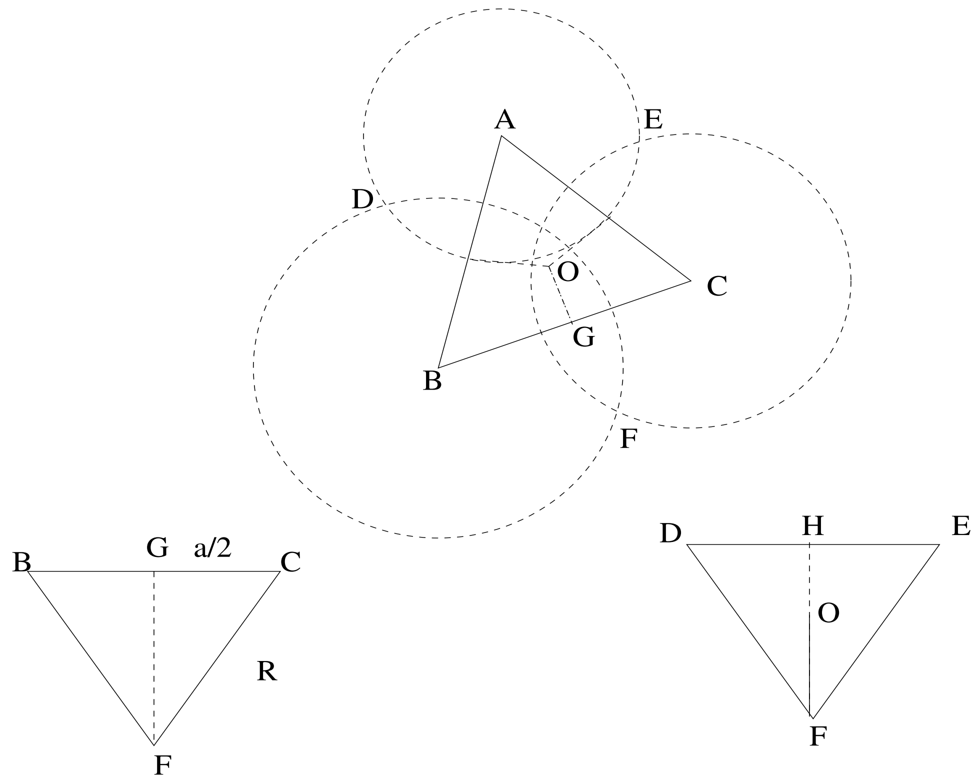

Figure 3: Diagram For Question 2

$$\overline{GF}^2 = R^2 - \frac{a^2}{4} \Rightarrow \overline{GF} = \sqrt{R^2 - \frac{a^2}{4}}$$
$$\overline{GA}^2 = a^2 - \frac{a^2}{4} \Rightarrow \overline{GA} = \frac{a\sqrt{3}}{2}$$
$$\overline{GO} = \frac{1}{3}\overline{GA} \Rightarrow \overline{GO} = \frac{a\sqrt{3}}{6}$$
$$\overline{FO} = \overline{FG} + \overline{GO} = \sqrt{R^2 - \frac{a^2}{4}} + \frac{a\sqrt{3}}{6}$$
$$\overline{FH} = \frac{3}{2}\overline{FO} = \frac{3}{2}\sqrt{R^2 - \frac{a^2}{4}} + \frac{a\sqrt{3}}{4}$$
$$\overline{FE} = \frac{2}{\sqrt{3}}\overline{FH} = \frac{3}{\sqrt{3}}\sqrt{R^2 - \frac{a^2}{4}} + \frac{a}{2} = \sqrt{3}\sqrt{R^2 - \frac{a^2}{4}} + \frac{a}{2}$$

3. (*Solution by Ms. Amy Palmgren, a 4th-6th finisher*)

Since any integer power of 2 is even, the number we are looking for must ends in 2,4,6 or 8. Suppose there exists a positive integer power of 2 that ends with four equal digits. Let's call this number $p = 2^n$. We note that $n \geq 10$. We have 4 cases to consider:

Case 1: p is of the form $\ldots 2222$. Then $p^{n-1} = \frac{p}{2} = \begin{cases} \ldots 1111 \\ \text{or} \\ \ldots 6111. \end{cases}$

Neither case is possible.

Case 2: p is of the form $\ldots 4444$. Then $p^{n-1} = \frac{p}{2} = \begin{cases} \ldots 2222 \\ \text{or} \\ \ldots 7222. \end{cases}$

The first situation reduces to *Case 1* which we already know is impossible. If $\frac{p}{2}$ is of the form $\ldots 7222$ then $p^{n-2} = \frac{p}{4}$ is of the form $\ldots 611$ which is not possible.

Case 3: p is of the form $\ldots 6666$. Then $p^{n-1} = \frac{p}{2} = \ldots 333$ which is impossible.

Case 4: p is of the form $\ldots 8888$. Then $p^{n-1} = \frac{p}{2} = \begin{cases} \ldots 4444 \\ \text{or} \\ \ldots 9444. \end{cases}$

The first situation reduces to *Case 2* which we already know is impossible.

If $\frac{p}{2}$ is of the form $\ldots 9444$ then $p^{n-2} = \frac{p}{4} = \begin{cases} \ldots 4722 \\ \text{or} \\ \ldots 9722. \end{cases}$

If $\frac{p}{4}$ is of the form $\ldots 4722$ then $p^{n-3} = \frac{p}{8} = \begin{cases} \ldots 2361 \\ \text{or} \\ \ldots 7361. \end{cases}$

Neither situation is possible.

If $\frac{p}{4}$ is of the form $\ldots 9722$ then $p^{n-3} = \frac{p}{8} = \begin{cases} \ldots 4861 \\ \text{or} \\ \ldots 9861. \end{cases}$

Again neither situation is possible.

All outcomes with the same last digit cannot come from 2^n. Repeatedly dividing by 2 leaves a non 2,4,6 or 8 last digit eventually.

4. (*Solution by Mr. Brandon Long, Mr. Long placed second in the 2004 Kettering Math Olympiad.*)

It is not possible to move all coins into a sector with 2004 moves. Moves can only be wasted in pairs. (for example, move a coin out of a region and then return the same coin to its original region) This means the process to get all coins into a region must have an even number of moves to be accomplished in 2004 moves.

Figure 4 divides the sectors. We note the final sector by X. Sectors denoted by E are *even*, meaning an even number of moves must be made to get the coin to X. Sectors denoted by O are *odd*, meaning an odd numbers of moves is required to get the coin to X. Since there are 5 *odd* sectors, a total of $5 \times 9 = 45$ coins requires an odd number of moves to reach X. This results in an odd number of moves being required. Making 2004, or any other even number of moves, an impossible solution.

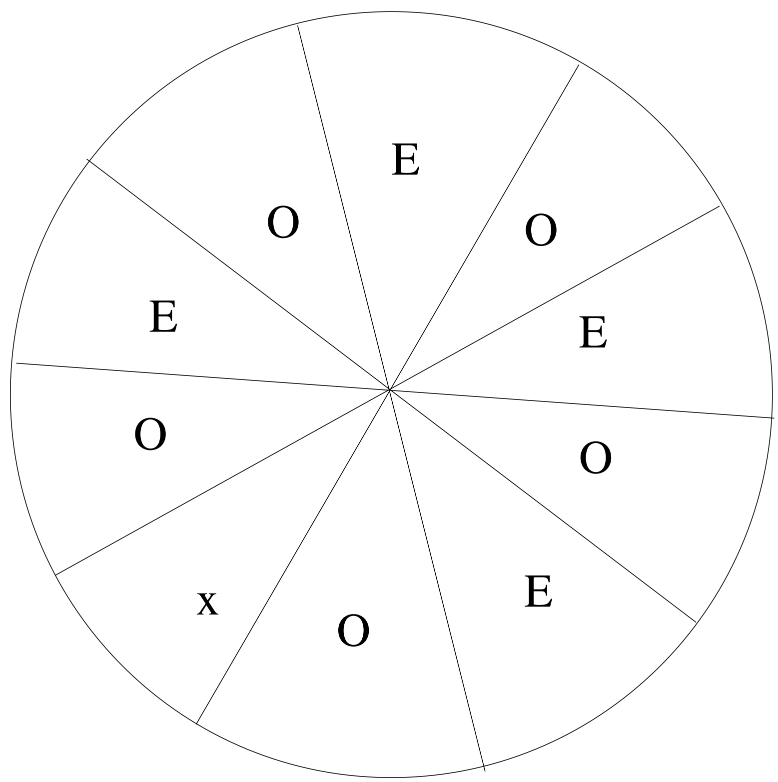

Figure 4: Diagram For Question 4

5. (*Solution by Mr. Dan Schultz, Mr. Schultz placed third in the 2004 Kettering Math Olympiad.*)

Yes, it is possible to divide a convex polygon with an arbitrary number of points inside into smaller convex polygons that each contain one point.

Lemma: Any rectangle is convex.
Proof: By definition all of its angles $= 90^0 < 180^0$.

We can draw an arbitrary convex polygon as in Figure 5. Let the arbitrary points be $P : \{p_1, p_2, \ldots, p_n\}$ have coordinates $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$.

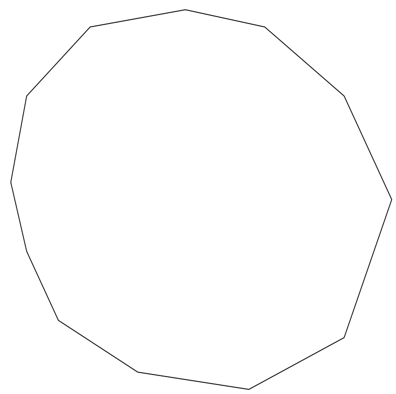

Figure 5: Diagram For Question 5

Now for $A, B \in \Re$ and $A \neq B$, there exists D such that $A < D < B$.

Group P into j sets with x coordinates the same and in increasing order. That is

$$\begin{aligned} P = \{ \qquad & (x_1, y_{1,1}), (x_1, y_{1,2}), \ldots, (x_1, y_{1,k_1}) \\ & (x_2, y_{2,1}), (x_2, y_{2,2}), \ldots, (x_2, y_{2,k_2}) \\ & \vdots \\ & (x_j, y_{j,1}), (x_j, y_{j,2}), \ldots, (x_j, y_{j,k_j})\} \end{aligned}$$

with $\sum_{l=1}^{j} = k_1 = n$ and $x_i < x_{i+1}$.

Next find $c_i's$ such that $x_i < c_i < x_{i+1}$. Now divide the polygon into *columns* with edges with the equation $x = c_i$.

Now we are left with columns with points inside that have different x coordinates. We can then divide these columns with horizontal lines in between the y coordinates of the points. If the column contains only one point then no further division is necessary.

6. (*Solution by Mr. Colin Clarke, Mr. Clark is the winner of the 2004 Kettering Math Olympiad.*)

It is possible for the grasshopper to visit every square. Let the grasshopper start from a point in the upper right square and then do a snake pattern between each of the squares as shown in Figure 6. Each time the grasshopper lands it is in the equivalent spot as in every other square. This path is valid from any point in the first square. The total amount of paint that the troll uses is $0.015 \times 64 = 0.96$ of a square. Since the grasshopper can start from any part of the first square, even if the troll paints the board to stop as many of these paths as possible, he will not be able to.

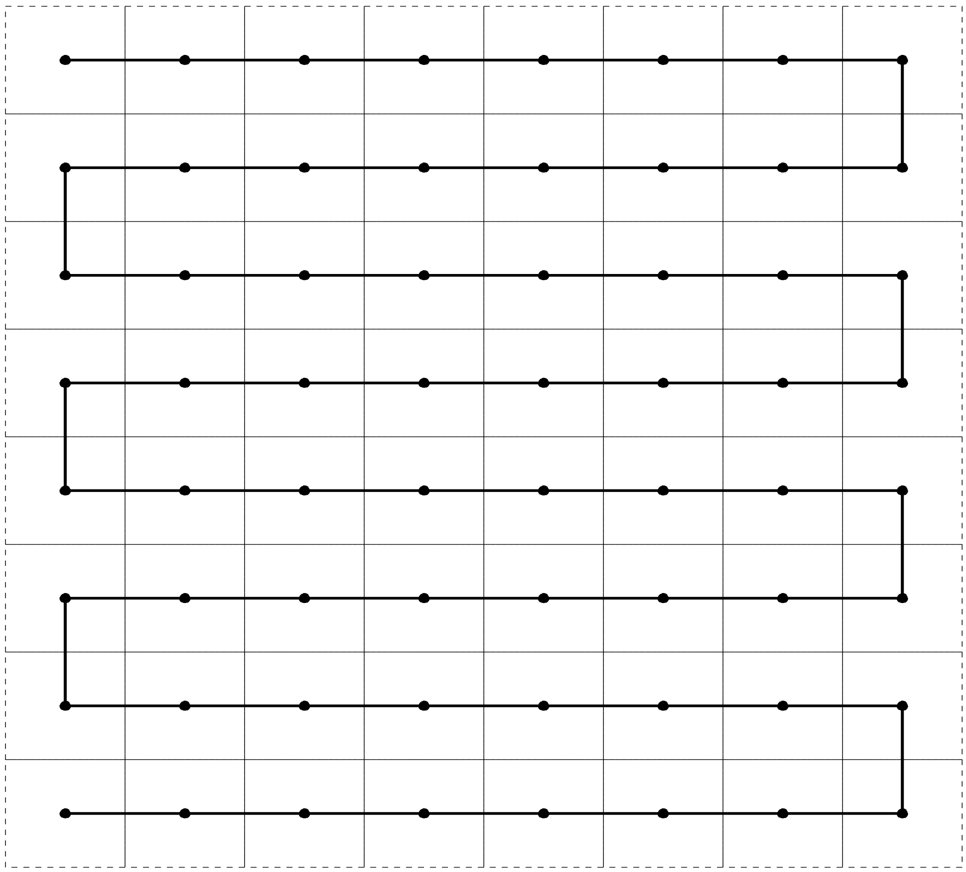

Figure 6: Diagram For Question 6