Kettering University Mathematics Olympiad For High School Students 2001, Sample Solutions

1. $x^2 - 2x + k = 0$ has at least one real root if

$$1 - k \geq 0 \quad \Longrightarrow \quad k \leq 1.$$

Thus the largest value for k is 1.

2. Yes. Here is one possible solution. Let I, G, F and S denote Indiana Jones, Indiana Jones' girlfriend, Indiana Jones' father and the side-kick to cross the bridge. Step 1: I and G cross the bridge, total time is 10 min.
Step 2: G returns to starting side of bridge, total time is 10 min.
Step 3: F and S cross the bridge, total time is 25 min.
Step 4: I returns to starting side of bridge, total time is 5 min.
Step 5: I and G cross the bridge, total time is 10m in.
Thus the total time is 60 min.

3. Take 1 coin from the first bag, two coins from the second bag, three coins from the third bag, ..., 10 coins from the tenth bag. The total number of coins taken from the bags is

$$1 + 2 + \ldots + 10 = \left(\frac{1+10}{2}\right) \times 10 = 55.$$

Weigh this 55 coins. If all coins were fare, the weight would be 550g. IF there were k coins from the bag withe counterfeit coins the weight is $(550 - k)$g. Therefore, subtracting from 550 the result of weighing one gets the number of the bag with counterfeit coins.

4. $\sqrt{x^2 + 4x + 4} = x^2 + 5x + 5$ implies $|x + 2| = x^2 + 5x + 5$. Thus we have two cases to consider:
Case 1: $x \geq -2$, so $|x + 2| = x + 2$.
Then

$$x + 2 = x^2 + 5x + 5 \quad \Longrightarrow x^2 + 4x + 3 = 0 \quad \Longrightarrow (x + 1)(x + 3) = 0.$$

Hence $x = -1$.
Case 1: $x < -2$, so $|x + 2| = -x - 2$.
Then

$$-x - 2 = x^2 + 5x + 5 \quad \Longrightarrow x^2 + 6x + 7 = 0 \quad \Longrightarrow x = -3 \pm \sqrt{2}.$$

Hence $x = -3 - \sqrt{2}$. That is, we have two possible values of x.

5. Let $O(6, 8)$ denote the centre of the circle. Then

$$|\overline{PO}|^2 = 6^2 + 8^2 = 100 \quad \Longrightarrow \quad |\overline{PO} = 10.$$

Triangle OAP and OBP are right angled triangles. Thus

$$|\overline{PA}|^2 = |\overline{PO}|^2 - |\overline{OA}|^2 = 100 - 25 \quad \Longrightarrow \quad |\overline{PA} = 5\sqrt{3}.$$

Let α, β denote the angle AOP and APO respectively. Then

$$\sin\beta = \frac{5}{10} \quad \Longrightarrow \beta = \frac{\pi}{6}, \alpha = \pi - \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}.$$

Let D denote the point on the circle and in the line segment OP. Then arc length $AD = DB = \frac{5\pi}{3}$, $ACB = 10\pi - 10\frac{\pi}{3} = \frac{20\pi}{3}$. Thus the path length is

$$2|\overline{PA}|^2 + \text{arclength}(ACB) = 10\sqrt{3} + \frac{20\pi}{3}.$$

6. (a) Area($ABCD$)=Area(ABD)+Area(BCD)

$$\begin{aligned} \text{Area}(ABD) &= \frac{1}{2}|AB| \cdot |AB| \cdot \sin A \leq \frac{|AB| \cdot |AD|}{2}, \\ \text{Area}(BDC) &= \frac{1}{2}|BC| \cdot |DC| \cdot \sin C \leq \frac{|BC| \cdot |DC|}{2}. \end{aligned}$$

Thus

$$\text{Area}(ABCD) \leq \frac{|AB| \cdot |AD|}{2} + \frac{|BC| \cdot |DC|}{2}.$$

(b) If $ABCD$ is not convex, then one of the diagonals is outside of $ABCD$. Let us assume that BD is outside.

Take the point C' such that CC' is perpendicular to BD and $CE = EC'$. Then $|BC| = |BC'|$ and $|CD| = |C'D|$.

The area of $ABCD$ is less than the area of the convex quadrilateral $ABC'D$. Therefore

$$\begin{aligned} \text{Area}(ABCD) &< \text{Area}(ABC'D) \\ &\leq \frac{|AB| \cdot |AD|}{2} + \frac{|BC'| \cdot |DC'|}{2} \\ &= \frac{|AB| \cdot |AD|}{2} + \frac{|BC| \cdot |DC|}{2}. \end{aligned}$$

(c) As it is shown in part b, it is sufficient to consider a convex quadrilateral.

Cut the quadrilateral by the diagonal AC. Let E be the midpoint of AC. Take a line which passes through E and is perpendicular to AC. Take point B' and F such that BB' is perpendicular to FE and $|BF| = |FB'|$. Then $|AB| = |B'C|$, $|AB'| = |BC|$ and Area($ABCD$)=Area($AB'CD$). From part a,

$$\text{Area}(AB'CD) \leq \frac{|AB'| \cdot |AD|}{2} + \frac{|BC'| \cdot |DC|}{2} = \frac{|BC| \cdot |AD|}{2} + \frac{|AB| \cdot |DC|}{2}.$$