PARAMETRIX OF THE MIXED PROBLEM FOR THE HYPERBOLIC SYSTEMS OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

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In the paper a parametrix for mixed problem for a hyperbolic system in a situation where the bicharacteristics are transversal to the boundary is constructed in the form of a global Fourier Integral Operator (FIO). So the construction of the parametrix of a mixed problem for second order hyperbolic equations, suggested by J.Chazarain, is extended to hyperbolic systems. The manifold of broken bicharacteristics, supplied with a structure of a homogeneous immersed Lagrangian submanifold, playes the role of the canonical relation for FIO. The symbol of FIO is constructed as an asymptotic sum of smooth sections of certain vector bundls on C.

§1. INTRODUCTION

In the present paper construction of the parametrix of the mixed problem for second order hyperbolic equation offered by J.Chazarain in [1-2] is extended to hyperbolic systems of first order partial differential equations. Following [1] and [2], we suppose that the bicharacteristics of the system are not tangent to the boundary. The parametrix is constructed in the form of Fourier Integral Operator (Maslov canonical operator). In §2 the canonical relation of the operator is described as the manifold of broken bicharacteristics. In §3 and §5 the symbol of the operator is found. The construction of the symbol is based on a smooth extension of canonical relation (§4). The main result of the paper is formulated in §6 and is proved in §7. A microlocal parametrix of the mixed problem in situations where the bicharacteristics may touch the boundary is constructed in [3-10].

Let X be a bounded domain in $\mathbf{R}^n$ with regular boundary ∂X. Denote $Q = \mathbf{R}_+ \times X$, $Q_0 = \{0\} \times X$, $\partial Q = \mathbf{R}_+ \times \partial X$. We consider the following distribution classes:

$$\mathcal{E}'_{r,\delta}(\partial Q) = \{g = (g_1, ..., g_r) \quad | \quad g_j \in \mathcal{E}'(\partial Q), \mathrm{singsupp}\, g_j \subset [\delta, +\infty) \times \partial X,\ j = 1, ..., r\};$$

$$\mathcal{D}'_{m,\delta}(Q_0) = \{u = (u_1, ..., u_m) \quad | \quad u_j \in \mathcal{D}\prime(Q_0),\ \mathrm{singsupp}\, u_j \subset Q_0,\ \mathrm{dist}(\mathrm{singsupp}\, u_j,$$

$$\partial Q) \geq \delta,\ j = 1, .., m\};$$

$$\hat{\mathcal{D}}'_m(Q) = \{u = (u_1, ..., u_m) \quad | \quad u_j \in \mathcal{D}'(Q) \text{ and there exists an extension } u_j \in \mathcal{D}'(\mathbf{R}^n)$$

$$\text{such that} \quad WF(u_j) \cap [N_{Q_0}(\mathbf{R}^{n+1}) \cup N_{\partial Q}(\mathbf{R}^{n+1})] = 0,\ j = 1, ..., m\}.$$

Notice, that all distributions from the class $\hat{\mathcal{D}}'_m(Q)$ have restrictions on Q_0 and ∂Q.

Let $A_j(x)$, $j = 0, 1, ..., n$ be smooth matrix-valued functions (of dimension $m \times m$) defined in some neighbourhood of $\overline{Q}$, $B(x)$ be a smooth matrix-valued function (of dimension $r \times m$), defined on ∂Q.

In the paper we consider the following mixed problem

$$Au = \frac{\partial u}{\partial x_0} - \sum_{j=1}^{n} A_j(x) \frac{\partial u}{\partial x_j} - A_0(x) u = 0 \quad \text{in} \quad Q, \tag{1.1}$$

$$u|_{Q_0} = u_0, \quad Bu|_{\partial Q} = g, \tag{1.2}$$

where $x = (x_0, x')$, $x_0 \in \mathbf{R}_+$, $x' = (x_1, ..., x_n) \in X$, $u_0 \in \mathcal{D}'_{m,\delta}(Q_0)$, $g \in \mathcal{E}'_{r,\delta}(\partial Q)$, and the solution u is searched in the class $\hat{\mathcal{D}}'_m(Q)$.

Definition 1.1. An operator E_δ is called parametrix of the problem (1.1) – (1.2) if

$$E_\delta : \mathcal{E}'_{r,\delta}(\partial Q) \times \mathcal{D}'_{m,\delta}(Q_0) \longrightarrow \hat{\mathcal{D}}'_m(Q) \tag{1.3}$$

and $E_\delta(g, u_0) - u \in C^\infty(\overline{Q})$, where u is the exact solution of the problem.

Our purpose is to construct the parametrix of the problem (1.1) – (1.2) in the form of global Fourier integral operator.

For $z = (x, \xi) \in T^*\overline{Q}$, denote by $\sigma_A(z)$ the principal symbol of the operator A, that is the matrix $\xi_0 I - \sum\limits_{j=1}^{n} A_j(x)\xi_j$, where I is the identity matrix and $\xi = (\xi_0, \xi')$, $\xi_0 \in \mathbf{R}$, $\xi' = (\xi_1, ..., \xi_n) \in \mathbf{R}^n$. Denote also $p(z) = \det \sigma_A(z)$. We assume that the system (1.1) is strictly hyperbolic, that is the following condition holds.

Condition 1. For each $x \in \overline{Q}$, $\xi' \in \mathbf{R}^n \setminus \{0\}$ the equation $p(x, \xi) = 0$ has m distinct real roots with respect to ξ_0.

We assume also that the boundary is non-characteristic, that is

Condition 2. For each z from the conormal bundle $N(\partial Q)$,

$$p(z) \neq 0.$$

Let $\pi_Q : T^*\overline{Q} \longrightarrow \overline{Q}$ be the natural projection. We use the following notation

$$\mathrm{Char} A = \{z \in T^*\overline{Q} \quad | \quad p(z) = 0\}$$

$$\mathrm{Char}_b A = \{z \in \mathrm{Char} A \quad | \quad \pi_Q z \in \partial Q, \ \frac{\partial p}{\partial \xi_0}(z) > 0\}.$$

Let $\Phi_s : \mathrm{Char}A \longrightarrow \mathrm{Char}A$ be the bicharacteristic flow i.e Φ_s is the flow of the Hamilton field H_p. Denote

$$\mathrm{Char}_b^+ A = \{z \in \mathrm{Char}_b A \quad | \quad \text{there exists } \varepsilon > 0 \text{ such that for any } s \in (0, \varepsilon),\ \pi_Q \Phi_s z \in Q\}$$

$$\mathrm{Char}_b^- A = \{z \in \mathrm{Char}_b A \quad | \quad \text{there exists } \varepsilon > 0 \text{ such that for any } s \in (0, \varepsilon),\ \pi_Q \Phi_s z \notin \overline{Q}\}.$$

Denote also by π_b the canonical projection $\pi_b : T^*\overline{Q}\big|_{\partial Q} \longrightarrow T^*(\partial Q)$.

Condition 3. The mapping

$$\pi_b : \mathrm{Char}_b^+ A \longrightarrow WF(g)$$

is $r-$ multiple covering.

Condition 3 means that from each point of the wave front set of g exactly r bicharacteristics enter the domain $\overset{\circ}{T}{}^*Q$.

Condition 4. Each bicharacteristic issueing from $\overset{\circ}{T}{}^*Q$ intersects the boundary of the domain Q transversally.

We also assume that the Lopatinski condition holds. To formulate it in a convenient form, let us consider the trivial vector bundle E^m on $\overset{\circ}{T}{}^*\overline{Q}$, where the fiber is $\mathbf{R}^m$. Because for each $z \in \overset{\circ}{T}{}^*\overline{Q}$ the matrix $\sigma_A(z)$ has m distinct real eigenvalues, the fiber on z is decomposed into direct sum of m invariant subspaces of the matrix $\sigma_A(z)$. Consider the restriction of that vector bundle on $\mathrm{Char}A$, that is the trivial vector bundle $\mathrm{Char}A \times \mathbf{R}^m \longrightarrow \mathrm{Char}A$. This vector bundle has the linear subbundle $K \longrightarrow \mathrm{Char}A$ in which the fiber over the point $z \in \mathrm{Char}A$ is $\mathrm{Ker}\,\sigma_A(z)$. Similarly consider the trivial vector bundle $E^m(\overset{\circ}{T}{}^*\partial Q) = \mathbf{R}^m \times \overset{\circ}{T}{}^*\partial Q \longrightarrow \overset{\circ}{T}{}^*\partial Q$ and the smooth trivial fibering $G_{m,r}(\overset{\circ}{T}{}^*\partial Q)$ where the fiber on the point $m \in \overset{\circ}{T}{}^*\partial Q$ is Grassman manifold of r–dimensional subspaces of $\mathbf{R}^m$. The Condition 3 implies that for every $z \in WF(g)$ we have that $\pi_b^{-1}(z) = (z_1, ..., z_r)$, where $z_j \in \mathrm{Char}_b^+ A$, $j = 1, ..., r$. Let $V(z)$ be a section of the fibering $G_{m,r}(\mathring{T}^* \partial Q)$ defined in the following way:

$$V(z) = K(z_1) + K(z_2) + ... + K(z_r),$$

where $\{z_1, ..., z_r\} = \pi_b^{-1}(z)$.

Condition 5. For each $z \in \mathring{T}^* \partial Q$, $\quad B \in \mathrm{Is}(V, E^r(\mathring{T}^* \partial Q))$.

§2. CANONICAL RELATION

Let

$$\mathrm{Ref}\, A = \{(z_+, z_-) | z_\pm \in \mathrm{Char}_b^\pm A, \pi_Q z_+ = \pi_Q z_-,\ z_+ - z_- = 0 \quad \text{on} \quad T_{\pi_Q z_+} \partial Q\}.$$

Denote

$$\mathrm{Char}_0\, A = \{z \in \mathrm{Char} A | \pi_Q z \in Q_0\}.$$

Definition 2.1. (see [1]). A mapping

$$\gamma : [s_0, s_1] \setminus J_\gamma \longrightarrow \mathring{T}^* Q,$$

where $J_\gamma = \{\widehat{s}_0, ..., \widehat{s}_{k(\gamma)}\}$ is a finite subset of $[s_0, s_1]$, is called broken bicharacteristic of the operator A if the following conditions hold:

(i) if I is an interval in $[s_0, s_1] \setminus J_\gamma$, then the curve $I \ni s \longmapsto \gamma(s)$ is a bicharacteristic of the operator A on Q;

(ii) $s_0 = \widehat{s}_0 < \widehat{s}_1 < ... < \widehat{s}_{k(\gamma)} < s_1$, $\gamma(s_0) \in \mathrm{Char}_0 A \cup \mathrm{Char}_b^+ A$;

(iii) there exist the limits $\gamma(\widehat{s}_0 + 0)$, $\gamma(\widehat{s}_j \pm 0)$ and

$$(\gamma(\widehat{s}_j + 0), \gamma(\widehat{s}_j - 0)) \in \mathrm{Ref} A, \tag{2.1}$$

where $j = 1, ..., k(\gamma)$.

Denote by C the set of all broken bicharacteristics and C_i the subset of those which have exactly i reflections from the boundary, that is $C_i = \{\gamma \in C | k(\gamma) = i\}$.

Let $\pi_0 : \overset{\circ}{T}{}^*\overline{Q}\Big|_{Q_0} \longrightarrow \overset{\circ}{T}{}^*Q_0$ be the canonical projection. Denote by π the mapping

$$\pi : (\mathrm{Char}_0 A \cup \mathrm{Char}_b^+ A) \longrightarrow \overset{\circ}{T}{}^*Q_0 \cup \overset{\circ}{T}{}^*(\partial Q)$$

such that $\pi|_{\mathrm{Char}_0 A} = \pi_0, \quad \pi|_{\mathrm{Char}_b^+ A} = \pi_b$. Consider also the mappings

$\kappa_0 : C_i \longrightarrow \mathrm{Char}_0 A \cup \mathrm{Char}_b^+ A, \quad \kappa_j : C_i \longrightarrow \mathrm{Char}_b^+ A,\ j = 1, ..., i, \quad \chi : C_i \longrightarrow \overset{\circ}{T}{}^*Q$

defined in the following way:

$$\kappa_j(\gamma) = \gamma(\widehat{s}_j + 0),\ j = 0, 1, ..., i,\ \chi(\gamma) = \gamma(s_1). \tag{2.2}$$

The subsets of C_i we call open sets if there are preimages of open sets with respect to these mappings. We introduce on $C_i, i = 0, 1, ...,$ the topology generated by these open subsets. These generate a topology on C.

Let us show that for each $\gamma \in C$ there exists a neighbourhood $U_\gamma \subset C$ such that the mapping $\chi|_{U_\gamma}$ is a bijection on the image. Indeed, let U_j be a neighbourhood of $\gamma(\widehat{s}_j + 0)$ such that the equality $\pi z_1 = \pi z_2$ implies $z_1 = z_2$ for each $z_1, z_2 \in U_j$. The existence of such neighbourhood is provided by the Condition 4. Now we can take

$$U_\gamma = \bigcap_{j=0}^{k(\gamma)} \kappa_j^{-1}(U_j).$$

Let us cover C by such neighbourhoods and take the compositions of χ and coordinate mappings of the manifold Char$A$ to be the coordinate mappings. So we obtain a C^∞–structure, which converts C into the $(2n+1)$ – dimensional manifold.

Denote by $\imath$ the following mapping

$$\imath : C \longrightarrow \mathrm{Char}A \times (\mathring{T}^* Q_0 \cup \mathring{T}^* \partial Q), \tag{2.3}$$

where

$$\imath(\gamma) = (\chi(\gamma), \pi \circ \kappa_0(\gamma)). \tag{2.4}$$

The manifold C has a stratification

$$C = C^1 \cup C^2 \cup C^3,$$

where $C^1 = \kappa_0^{-1}(\mathrm{Char}_b^+ A)$, $C^2 = \kappa_0^{-1}(\mathrm{Char}_0 A)$, $C^3 = (\pi_Q \circ \kappa_0)^{-1}(\overline{\partial Q} \cap \overline{Q}_0)$.

It is easy to see that C^1 and C^2 are $(2n+1)$–dimensional and C^3 is $2n$–dimensional manifolds.

Let us consider sympletic manifolds $\mathring{T}^* Q \times \mathring{T}^*(\partial Q)$ and $\mathring{T}^* Q \times \mathring{T}^*(Q_0)$ with canonical 2-forms $\omega_b = \sigma_{T^*Q} - \sigma_{T^*(\partial Q)}$ and $\omega_0 = \sigma_{T^*Q} - \sigma_{T^*Q_0}$ correspondingly.

Theorem 2.1 $(C_1, \imath)$ and $(C_2, \imath)$ are homogeneous immersed Lagrangian submanifolds in $\mathring{T}^* Q \times \mathring{T}^*(\partial Q)$ and $\mathring{T}^* Q \times \mathring{T}^*(Q_0)$ correspondingly.

Proof. Let us prove the theorem for the manifold $(C_1, \imath)$ (the proof for the $(C_2, \imath)$ is similar). According to the definition of ω_b we have

$$\imath^* \omega_b = \chi^* \sigma_{T^*Q} - (\pi \circ \kappa_0)^* \sigma_{T^* \partial Q}. \tag{2.5}$$

It is easy to see that each point $\gamma_0 \in C_1$ has a neighbourhood U_{γ_0} such that the mapping

$$\imath_0 = \pi \circ \kappa_0 \,:\, U_{\gamma_0} \longrightarrow \pi \circ \kappa_0(U_{\gamma_0})$$

is a diffeomorphism. Hence

$$\imath^*\omega_b = \imath_0^*(\imath_0^{-1}\chi^*\sigma_{T^*Q} - \sigma_{T^*\partial X}). \tag{2.6}$$

Denote

$$\widetilde{\Phi}|\,\imath_0(U_{\gamma_0}) \longmapsto \chi(U_{\gamma_0})$$

the mapping such that $\widetilde{\Phi} = \chi \circ \imath_0^{-1}$. We show that $\widetilde{\Phi}$ is a simplexomorphism that is

$$\widetilde{\Phi}^*\sigma_{T^*Q} - \sigma_{T^*\partial Q} = 0. \tag{2.7}$$

$\widetilde{\Phi}z_0 = z$ implies the existence of such $\gamma \in U_{\gamma_0}$ that $\kappa_0(\gamma) = z_0$, $\chi(\gamma) = z$. The neighbourhood U_{γ_0} may be chosen in such a way that for each $\gamma \in U_{\gamma_0}$, $k(\gamma) = k(\gamma_0)$. In case $k(\gamma_0) = 0$ we have $\widetilde{\Phi} = \Phi_{s_1 - s_0}$ and the equality (2.7) follows from the fact that Φ_s is a simplexomorphism. Let us show that this property is valid also in the presence of reflections $(k(\gamma_0) > 0)$. Let

$$S_j^{\pm} = \{z \in \mathrm{Char}_b^{\pm} A| \quad \text{there exists } \gamma \in U_{\gamma_0} \text{ such that } z = \gamma(\widehat{s}_j \pm 0)\}. \tag{2.8}$$

Let also

$$r_j \,:\, S_j^- \longrightarrow S_j^+,$$

where $r_j(z_-) = z_+$, if there exists $\gamma \in U_{\gamma_0}$ such that $z_\pm = \gamma(\widehat{s}_j \pm 0)$.

Let $k(\gamma_0) = k > 0$, then

$$\widetilde{\Phi} = \Phi_{s_1 - \widehat{s}_k} \circ r_k \circ \Phi_{\widehat{s}_k - \widehat{s}_{k-1}} \circ r_{k-1} \circ ... \circ r_1 \circ \Phi_{\widehat{s}_1 - s_0}. \tag{2.9}$$

To prove that $\widetilde{\Phi}$ is a simplexomorphism it is enough to show that r_j, $j = 1, 2, ..., k$ have the same property. In small enough neighbourhood V_j of the point $\pi_Q\gamma(\widehat{s}_j+0)$ let us choose such a coordinate system that the boundary of the domain Q is written as $x_n=0$. Let $\xi=(\xi_0,...,\xi_n)$ be the dual coordinates, then

$$r_j(x_0,x_1,...,x_{n-1},0,\xi_0,\xi_1,...,\xi_{n-1},\xi_n)=(x_0,x_1,...,x_{n-1},0,\xi_0,\xi_1,...,\xi_{n-1},\widetilde{\xi}_n) \tag{2.10}$$

and

$$\sigma=\sum_{j=0}^{n}d\xi_j\wedge dx_j.$$

We mention that on $TS_j^{\pm}$ the form dx_n is zero, so

$$r_j^*\sigma-\sigma=0. \tag{2.11}$$

From (2.5) and (2.6) one concludes that $(C_1,\imath)$ is Lagrangian. Now we shall prove that it is homogeneous.

Define on C the action of the group $\mathbf{R}_+$ generated by the action of that group on $T^*\overline{Q}$. For each pair $\gamma\in C$ and $\rho\in\mathbf{R}_+$ we put into the correspondence the bicharacteristic $\rho\gamma$ in the following way

$$\left[\rho^{1-m}\widehat{s}_0,\rho^{1-m}\widehat{s}_1\right]\setminus\{\rho^{1-m}\widehat{s}_0,...,\rho^{1-m}s_{k(\gamma)}\}\ni s\longmapsto\rho\gamma=\rho_0\gamma(s)\in\mathring{T}^*Q. \tag{2.12}$$

To verify that $\rho\gamma$ is a broken bicharacteristic it is sufficient to notice that the Hamilton-Jacobs system and the condition (2.1) are invariant with respect to the change of variables:

$$s\longmapsto\rho^{1-m}s,\ x\longmapsto x,\ \xi\longmapsto\rho\xi.$$

According to the definition of the action of $\mathbf{R}_+$ on C

$$\imath \circ \rho = \rho \circ \imath. \tag{2.13}$$

Let ζ be the vector field on C_1 generated by the action of $\mathbf{R}_+$. The equality (2.13) implies that the vector fields $d\chi(\zeta)$ and $d(\pi \circ \kappa_0)\zeta$ are generated by the action of $\mathbf{R}_+$ on T^*Q and $T^*\partial Q$. Therefore

$$\imath^*\alpha = \chi^*\alpha_{T^*Q} - (\pi \circ \kappa_0)^*\alpha_{T^*\partial Q} = \chi^*(d\chi\zeta \rfloor \sigma_{T^*Q}) - (\pi \circ \kappa_0)^*(d(\pi \circ \kappa_0)\zeta \rfloor \sigma_{T^*\partial Q}) = 0$$

which means that C_1 is homogeneous Lagrangian manifold.

§3. PRINCIPAL SYMBOL OF THE FIO F_0

Let $\Omega^{1/2}(\mathrm{Char}A)$ be the linear bundle of semidensities on $\mathrm{Char}A$. Denote by K_Ω the linear bundle which is the tenzor product of the bundles K and $\Omega^{1/2}$. Let $P(z)$ be the projector

$$P(z) : \mathbf{R}^m \longrightarrow \mathrm{Ker}\sigma_A(z),$$

generated by the decomposition of $\mathbf{R}^m$ into a direct product of invariant subspaces of the matrix $\sigma_A(z)$. Correspondingly

$$P_\Omega : \mathbf{R}^m \bigotimes \Omega^{1/2}(\mathrm{Char}A) \longrightarrow K_\Omega$$

will be the homomorphism of vector bundles generated by the projector $P(z)$.

For any point $z_0 \in \mathrm{Char}A$ there exist a neighborhood $U_{z_0} \subset T^*Q$ and smooth function $\lambda(z)$ defined on U_{z_0} such that for every $z \in U_{z_0}$, $\lambda(z)$ is the eigenvalue of $\sigma_A(z)$ vanishing at z_0. So we define a smooth function $\lambda(z)$ vanishing on the

Char$A$. Let v be a smooth section of the linear bundle K_Ω. We define a Poisson bracket

$$\{\sigma_A - \lambda I, v\} = \sum_{j=0}^{n} [(\sigma_A - \lambda I)_{\xi_j} v_{x_j} - (\sigma_A - \lambda I)_{x_j} v_{\xi_j}]. \tag{3.1}$$

To verify the correctness of this definition let us show that the section v on the right hand side of (3.1) is differentiated by directions tangent to Char$A$. Let $v(z)$ be an eigenvector of $\sigma_A(z)$ corresponding to eigenvalue $\lambda(z)$, smoothly depending on z in a neighborhood of the point $z_0 \in$ Char$A$. After differentiation of the equality $(\sigma_A - \lambda I)v = 0$ with respect to ξ_0 and applying $(\sigma_A)_{\xi_0} = I$ we obtain

$$v + \sigma_A v_{\xi_0} - \lambda v_{\xi_0} - \lambda_{\xi_0} v = 0.$$

Now let us apply the projector $P(z)$ to both sides of the last equality. Since $P(z)\sigma_A(z) = 0$ we have that $\lambda_{\xi_0} = 1$ on Char$A$. Let us differentiate now the equality $(\sigma_A - \lambda I)v = 0$ with respect to ξ_0 and x_0. We obtain the following relation

$$(\sigma_A - \lambda I)\, v_{\xi_0 x_0} + (\sigma_A - \lambda I)_{\xi_0} v_{x_0} + (\sigma_A - \lambda I)_{x_0} v_{\xi_0} - \lambda_{x_0 \xi_0} = 0.$$

Applying $P(z)$ to this equality we show that

$$(\sigma_A - \lambda I)_{x_0} v_{\xi_0} = \lambda_{x_0 \xi_0} v \tag{3.2}$$

on Char$A$. For $i = 1, .., n$ in same way we obtain

$$(\sigma_A - \lambda I)_{x_i} v_{\xi_0} = \lambda_{x_i \xi_0} v, \tag{3.3}$$

$$(\sigma_A - \lambda I)_{\xi_i} v_{\xi_0} = \lambda_{\xi_i \xi_0} v. \tag{3.4}$$

Since the vector field H_λ is tangent to Char$A$, we have

$$\{\lambda, \sigma_A\} v_{\xi_0} = \{\lambda, \lambda_{\xi_0}\} v = (H_\lambda \lambda_{\xi_0}) v = 0. \tag{3.5}$$

Therefore, on Char$A$

$$\{\sigma_A, v\} = \{\sigma_A, v\} + \{\lambda, \sigma_A\} v_{\xi_0} = \sum_{j=0}^{n} (\sigma_A)_{\xi_j} (\lambda_{\xi_0} v_{x_j} - \lambda_{x_j} v_{\xi_0}) -$$

$$- \sum_{j=1}^{n} (\sigma_A)_{x_j} \left(\lambda_{\xi_0} v_{\xi_j} - \lambda_{\xi_j} v_{\xi_0}\right). \tag{3.6}$$

The 1–form $d\lambda$ vanishes in the vector fields $\lambda_{\xi_0} \frac{\partial}{\partial x_j} - \lambda_{x_j} \frac{\partial}{\partial \xi_0}$ and $\lambda_{\xi_0} \frac{\partial}{\partial \xi_j} - \lambda_{\xi_j} \frac{\partial}{\partial \xi_0}$, $j = 1, ..., n$, so they are tangent to Char$A$. To complete the proof we can notice that $\{\lambda I, v\} = H_\lambda v$ and recall that H_λ is tangent to Char$A$.

Now let us consider a non-vanishing smooth section $v(z)$ of K_Ω in a neighborhood U_{z_0} of a point $z_0 \in$ Char$A$. We define a function $\mu(z)$ on U_{z_0} by

$$\mu(z) v(z) = P(z) \left[\sigma_1(z) v(z) + \{\sigma_A - \lambda I, v\}\right], \tag{3.7}$$

where $\sigma_1(z)$ is the subsymbol of A, that is

$$\sigma_1(z) = \frac{1}{2} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} A_j - A_0 = \frac{1}{2} \sigma(A + A^*). \tag{3.8}$$

Actually $\mu(z)$ is independent of the choice of the local section $v(z)$ and thus it is invariantly defined on Char$A$. In other words, $\mu(z)$ is invariant with respect to transformation $v(z) \longmapsto \rho(z) v(z)$, where $\rho(z)$ is arbitrary smoth scalar function given on U_{z_0}. To show this, we observe that on Char$A$

$$P(z) \left[\sigma_1(z) \rho(z) v(z) + \{\sigma_A - \lambda I, \rho v\}\right] - \rho(z) P(z) \left[\sigma_1(z) v(z) + \{\sigma_A - \lambda I, v\}\right] =$$

$$= \sum_{j=1}^{n} \left[\rho_{x_j} P(z) (\sigma_A - \lambda I)_{\xi_j} v - \rho_{\xi_j} P(z) (\sigma_A - \lambda I)_{x_j} v\right] =$$

$$= \sum_{j=1}^{n} \rho_{x_j} P(z) \left[\lambda_{\xi_0} (\sigma_A - \lambda I)_{\xi_j} - \lambda_{\xi_j} (\sigma_A - \lambda I)_{\xi_0} \right] v(z) -$$

$$- \sum_{j=0}^{n} \rho_{\xi_j} P(z) \left[\lambda_{\xi_0} (\sigma_A - \lambda I)_{x_j} - \lambda_{x_j} (\sigma_A - \lambda I)_{\xi_0} \right] v(z).$$

To show that the last sum is zero, it is sufficient to apply the vector fields

$$\lambda_{\xi_0} \frac{\partial}{\partial \xi_j} - \lambda_{\xi_j} \frac{\partial}{\partial \xi_0}, \quad j = 0, 1, ..., n,$$

and

$$\lambda_{\xi_0} \frac{\partial}{\partial x_j} - \lambda_{x_j} \frac{\partial}{\partial \xi_0}, \quad j = 0, 1, ..., n$$

to both sides of $(\sigma_A - \lambda I)v = 0$ and then to act by the projector $P(z)$. We define also a function μ_C on C in the following way:

$$\mu_C(\gamma) = \int_{s_0}^{s_1} \mu(\gamma(s))\, ds.$$

Now we are going to define a linear connection $\bigtriangledown$ on $\mathbf{K}_\Omega$.

Let η be any vector field on $\mathrm{Char}A$, $v(z)$ be any section of K_Ω, considered as a section of the trivial vector bundle

$$\Omega^{1/2}(\mathrm{Char}A) \times \mathbf{R}^m \longrightarrow \mathrm{Char}A.$$

We define

$$\bigtriangledown_\eta v(z) = P(z) \left[D_\eta v(z) \right], \tag{3.9}$$

where D_η is a Lie derivative of v along the vector field η. In the case of the constant coefficients this connection has a simple geometrical meanings: parallel transfer of a vector belonging to a fiber of K_Ω along some curve γ consists of a transfer along γ in the trivial vector bundle $\Omega^{1/2}(\mathrm{Char}A) \bigotimes \mathbf{R}^m \longrightarrow \mathrm{Char}A$ and of projection on K_Ω.

Let $\gamma(z_1, z_2)$ be a bicharacteristic connecting $z_1, z_2 \in T^*\overline{Q}$. Denote by $R^{\bigtriangledown, K}_{z_1, z_2}$ an operator of parallel transfer generated by $\bigtriangledown$ along the curve $\gamma(z_1, z_2)$ in the vector bundle K_Ω. So if $v_1 \in K_{z_1}$ then $R^{\bigtriangledown, K}_{z_1, z_2} v_1 = v_2 \in K_{z_2}$ and there is a section s of K_Ω for which $\bigtriangledown s = 0, \; s(z_1) = v_1, \, s(z_2) = v_2$.

Consider vector bundle $\mathcal{M}^r_K = K \times K \times ... \times K$ (r factors) on Char$A$. The fiber of $\mathcal{M}^r_K$ is r–dimensional vector space of $m \times r$ matrices with columns belonging to K. So the operator $R^{\bigtriangledown, K}_{z_1, z_2}$ acting in K_Ω generates an operator of parallel transfer $R^{\bigtriangledown}_{z_1, z_2}$ acting in the vector bundle $\mathcal{M}^r_K$.

Assuming $z_o \in \mathring{T}^* \partial Q$, from the condition 3 we have $\pi_b^{-1} z = \{z_1, ..., z_r\}$, where $z_j \in \mathrm{Char}^+_b A, \; j = 1, ..., r$. There exists a neigborhood U_{z_0} of z_0 such that there are smooth non-vanishing sections $v_i(z)$ of linear vector bundle K_Ω defined in neighborhoods $U_{z_i^0}$ of the points z_i^0 and $U_{z_0} \subset \pi_b U_{z_i^0}$ for $i = 1, ..., r$. Thus one can define smooth matrix-valued function $V(z)$ in U_{z_0}, whose columns are $v_1(z), ..., v_r(z)$.

Now we define the following matrix functions in U_{z_0} (see [13])

$$W(z) = V(z) \left[B(z) V(z) \right]^{-1}. \tag{3.10}$$

We can do this because Condition 5 implies that $B(z)V(z)$ is nondegenerate matrix. $W(z)$ does not depend on the choice of $V(z)$, so in this way we can define it on $\mathring{T}^* \partial Q$. Consider the following sections Ω_0 and Ω of $\mathcal{M}^r_K$:

$$\Omega_0(z) = P(z) W(\pi_b z), \tag{3.11}$$

$$\Omega(z) = -P(z) W(\pi_b z) B(\pi_b z). \tag{3.12}$$

Denote by $\mathcal{M}^r_K(C_1)$ the vector bundle obtained by pull back of fibers of $\mathcal{M}^r_K$

by means of the mapping $\chi : C_1 \longrightarrow \mathrm{Char} A$. We define the principal symbol of $\mathcal{F}_0$ to be a smooth section of the vector bundle $\mathcal{M}_K^r(C_1) \bigotimes \Omega^{1/2}(C_1)$.

If $\nu_{\partial Q}$ is any non-vanishing section of the linear bundle $\Omega^{1/2}(\partial Q) \longrightarrow \partial Q, \ \gamma \in C_1$, then we define the principal symbol $\sigma_{\mathcal{F}_0}$ by the formula

$$\sigma_{\mathcal{F}_0}(\gamma) = \chi^* \exp\left[-\mu_C(\gamma)\right] \mathcal{R}^{\nabla}_{\kappa_{k(\gamma)-1}(\gamma),\chi(\gamma)} \Omega(\kappa_{k(\gamma)-1}(\gamma)) \circ$$

$$\circ \mathcal{R}^{\nabla}_{\kappa_{k(\gamma)-2}(\gamma),\kappa_{k(\gamma)-1}(\gamma)} \Omega(\kappa_{k(\gamma)-2}(\gamma)) \ldots \circ \mathcal{R}^{\nabla}_{\kappa_1(\gamma),\kappa_2(\gamma)} \Omega(\kappa_1(\gamma)) \circ$$

$$\circ \mathcal{R}^{\nabla}_{\kappa_0(\gamma),\kappa_1(\gamma)} (\Omega_0(\kappa_0(\gamma)) \bigotimes \pi_Q^* |dx|^{1/2} \bigotimes \pi_Q^* \nu_{\partial Q}). \tag{3.13}$$

Notice that one can correctly define the pull back of semidensities from Q and ∂Q to $\mathrm{Char} A$ and $\mathrm{Char}_b^+ A$ with respect to π_Q and from $\mathrm{Char} A$ to C with respect to χ, because the mappings $\pi_Q : \ \mathrm{Char} A \longrightarrow Q, \quad \pi_Q : \ \mathrm{Char}_b^+ A \longrightarrow \partial Q$ and $\chi : \ C \longrightarrow \mathrm{Char} A$ are surjective.

In the same way, we denote by $\mathcal{M}_K^m(C_2)$ the vector bundle on C_2 obtained by means of pull back of the fibers of vector bundle $\mathcal{M}_K^m$ on $\mathrm{Char} A$ with respect to mapping $\chi : \ C_2 \longrightarrow \mathrm{Char} A$. Correspondingly, for $\gamma \in C_2$ we define $\sigma_{\mathcal{F}_0(\gamma)}$ as a smooth section of vector bundle $\mathcal{M}_K^m(C_2) \bigotimes \Omega^{1/2}(C_2)$:

$$\sigma_{\mathcal{F}_0}(\gamma) = \chi^* \exp\left[-\mu_C(\gamma)\right] \mathcal{R}^{\nabla}_{\kappa_{k(\gamma)-1}(\gamma),\chi(\gamma)} \Omega(\kappa_{k(\gamma)-1}(\gamma)) \circ$$

$$\circ \mathcal{R}^{\nabla}_{\kappa_{k(\gamma)-2}(\gamma),\kappa_{k(\gamma)-1}(\gamma)} \Omega(\kappa_{k(\gamma)-2}(\gamma)) \ldots \circ \mathcal{R}^{\nabla}_{\kappa_1(\gamma),\kappa_2(\gamma)} \Omega(\kappa_1(\gamma)) \circ$$

$$\circ \mathcal{R}^{\nabla}_{\kappa_0(\gamma),\kappa_1(\gamma)} (\Omega(\kappa_0(\gamma)) \bigotimes \pi_Q^* |dx|^{1/2} \bigotimes \pi_Q^* dx'). \tag{3.14}$$

§4. EXTENDED CANONICAL RELATION

Denote by $\widetilde{Q}$ an open neighborhood of $\overline{Q}$, where coefficients of A are defined.

Correspondently

$$\widetilde{\mathrm{Char}}^{\pm} A = \{z \in \overset{\circ}{T}{}^{*}\widetilde{Q} | \quad p(z) = 0, \ \pm\frac{\partial p}{\partial \xi_0} > 0\}. \tag{4.1}$$

Let $\mathrm{Bich}^{\pm} A$ be the set of the bicharacteristics of A directed up $(+)$ or down $(-)$ with respect to the axes t so $\mathrm{Bich} A$ is the set of such mappings γ that

$$(-\infty, \infty) \supset [s_0, s_1] \ni s \longmapsto \gamma(s) \in \widetilde{\mathrm{Char}}^{\pm} A, \ s_0 < s_1.$$

Consider mappings $\beta_j \ : \quad \mathrm{Bich}^{\pm} A \longrightarrow \widetilde{\mathrm{Char}}^{\pm} A$, $j = 0, 1$ such that $\beta_j(\gamma) = \gamma(s_j)$. We allot $\mathrm{Bich}^{\pm} A$ by the structure of $(2n+2)$–dimensional C^{∞}–manifold, generated by

$$\beta = \beta_0 \times \beta_1 \ : \quad \mathrm{Bich}^{\pm} A \longrightarrow \widetilde{\mathrm{Char}}^{\pm} A \times \widetilde{\mathrm{Char}}^{\pm} A.$$

Denote

$$\mathrm{Bich}^{\pm}_b A = \{\gamma \in \mathrm{Bich}^{\pm} A | \quad \beta_0(\gamma) \in \mathrm{Char}^{+}_b A\},$$

$$\mathrm{Bich}_b A = \mathrm{Char}^{+}_b A \cup \mathrm{Bich}^{+}_b A \cup \mathrm{Bich}^{-}_b A.$$

In order to extend the mappings β_0, β_1, β to $\mathrm{Bich}_b A$ we extend them to $\mathrm{Char}^{+}_b A$ in letting:

$\beta_j|_{\mathrm{Char}^{+}_b A} = \mathrm{id}$, $\beta = \beta_0 \times \beta_1$. In addition $\mathrm{Bich}_b A$ is alloted by the structure of a smooth $(2n+1)$–dimensional manifold.

Definition 4.1. The mapping

$$\gamma|[s_0, s_1] \setminus J \ni s \longrightarrow \gamma(s) \in \mathrm{Char} A,$$

we call a broken bicharacteristic of A extended up to the boundary if the following conditions hold:

1) for any $\widetilde{s}_1 \in (s_0, s_1) \setminus J$ the restriction of γ to $[s_0, \widetilde{s}_1]$ is a broken bicharacteristic of A;

2) $\chi(\gamma) = \gamma(s_1 - 0) \in \mathrm{Char}_b^- A$.

Denote by $\hat{C}$ the set of all broken bicharacteristics of A, extended up to the boundary. On $\hat{C}$ we introduce a topology which is a preimage of a topology on $\mathrm{Char}_b^+ A$ with respect to mappings $\kappa_0, ..., \kappa_{k(\gamma)}, \chi$. Then we allot $\hat{C}$ by the structure of $2n$–dimensional C^∞–manifold by means of χ.

Consider submanifolds of the manifold $(\mathrm{Char}_b^+ A \cup \hat{C}) \times \mathrm{Bich}_b A$ defined in the following way:

$$\widetilde{C}_{1,i} = \{(\gamma_1, \gamma_2) | \quad \gamma_1 \in \mathrm{Char}_b^+ A,\ \gamma_2 \in \mathrm{Char}_b^+ A \cup \mathrm{Bich}_b^+ A \text{ for } i = 1 \text{ and}$$

$$\gamma_2 \in \mathrm{Bich}_b^- A \text{ for } i = 2,\ \gamma_1 = \beta_0(\gamma_2)\},$$

$$\widetilde{C}_{2,i} = \{(\gamma_1, \gamma_2) | \quad \gamma_1 \in \hat{C},\ \gamma_2 \in \mathrm{Char}_b^+ A \cup \mathrm{Bich}_b^+ A \text{ for } i = 1 \text{ and}$$

$$\gamma_2 \in \mathrm{Bich}_b^- A \text{ for } i = 2,\ (\beta_0(\gamma_2), \chi(\gamma_1)) \in \mathrm{Ref} A\}.$$

Let $\widetilde{C} = \widetilde{C}_{11} \cup \widetilde{C}_{12} \cup \widetilde{C}_{21} \cup \widetilde{C}_{22}$. From the condition 4 one can see that $\widetilde{C}$ is $(2n+1)$–dimensional C^∞–manifold and $C \subset \widetilde{C}$. Notice that

$$\widetilde{C} = \widetilde{C}^1 \cup \widetilde{C}^2 \cup \widetilde{C}^3 \quad \text{and} \quad C^i \subset \widetilde{C}^i,\ i = 1, 2, 3.$$

Consider the mapping $\tau : \widetilde{C} \longrightarrow \mathring{T}^*(\widetilde{Q}) \cup \mathring{T}^*(\partial Q) \times \mathring{T}^*(Q_0))$, defined by the formula $\tau = \beta_1 \times \pi \circ \kappa_0$, where κ_0 is extended to $\mathrm{Char}_b^+ A$ as an identity mapping.

As in the proof of the Lemma we show that $(\widetilde{C}_1, \tau)$ and $(\widetilde{C}_2, \tau)$ are immersed homogeneous Lagrangian submanifolds in $\mathring{T}^*(\widetilde{Q}) \times \mathring{T}^* \partial Q)$ and $\mathring{T}^*(\widetilde{Q}) \times (\mathring{T}^*(Q_0)$. We call $(\widetilde{C}, \tau)$ the extended canonical relation. As there is a closure of C in $\widetilde{C}$

then we can define

$$\partial C = \{(\gamma_1, \gamma_2) \in \widetilde{C}_{1,1} \cup \widetilde{C}_{2,1} | \quad \beta_1(\gamma_2) \in \overset{\circ}{T}{}^* \overline{Q}\Big|_{\partial Q}\}.$$

The boundary ∂C one can represent as a union of the non-intersecting submanifolds

$$\partial C = \partial_0 C \cup \partial_+ C \cup \partial_- C, \tag{4.2}$$

where

$$\partial_0 C = \{(\gamma_1, \gamma_2) \in \widetilde{C}_{1,1} | \quad \gamma_1 = \gamma_2\} \sim \mathrm{Char}_b^+ A,$$

$$\partial_+ C = \partial C \cap \widetilde{C}_{2,1} = \{(\gamma_1, \gamma_2) | \quad \gamma_1 \in \hat{C}, \gamma_2 \in \mathrm{Char}_b^+ A, \ (\chi(\gamma_1), \gamma_2) \in \mathrm{Ref} A\},$$

$$\partial_- C = \{(\gamma_1, \gamma_2) \in \partial C | \quad \gamma_2 \in \mathrm{Bich}_b^+ A\} \sim \hat{C}.$$

It follows from Condition 3 that $\partial_+ C$ is r–multiple covering over $\partial_- C$ with respect to the mapping

$$\partial_+ C \ni (\gamma_1, \gamma_2) \longmapsto \gamma_1 \in \hat{C} \sim \partial_- C.$$

Our purpose is to extend FIO $\mathcal{F}_0$ to FIO $\widetilde{\mathcal{F}}_0$ with the canonical relation $\widetilde{C}$. So we have to extend the principal symbol of $\mathcal{F}_0$ from C to $\widetilde{C}$. First let us consider the linear bundle $\widetilde{K}_\Omega \longmapsto \widetilde{\mathrm{Char}} A$, defined in the same way as $K_\Omega \longmapsto \mathrm{Char} A$. So one can extend the function $\mu(z)$ from $\mathrm{Char} A$ to $\widetilde{\mathrm{Char}} A$ and linear connection $\bigtriangledown$ from K to $\widetilde{K}$. Correspondingly we define an operator of parallel transfer $R^{\bigtriangledown, \widetilde{K}}_{z_1, z_2}$. Now we define the principal symbol of $\widetilde{\mathcal{F}}_0$ in the following way:

1) if $(\gamma_1, \gamma_2) \in C_{1,1} \cup C_{1,2}$, then

$$\sigma_{\widetilde{\mathcal{F}}_0}(\gamma_1, \gamma_2) = R^{\bigtriangledown, \widetilde{K}_\Omega}_{\beta_1(\gamma_2), \beta_0(\gamma_2)} \Omega_0(\pi_b \gamma_1) \bigotimes \beta_1^* |dx|^{1/2} \bigotimes \beta_0^* \nu_{\partial Q}. \tag{4.3}$$

2) if $(\gamma_1,\gamma_2)\in \widetilde{C}_{2,1}\cup\widetilde{C}_{2,2}$, then

$$\sigma_{\widetilde{\mathcal{F}}_0}(\gamma_1,\gamma_2)=R^{\nabla,\widetilde{K}_\Omega}_{\beta_1(\gamma_2),\beta_0(\gamma_2)}\Omega(\pi_b\beta_0(\gamma_2))\sigma_{\mathcal{F}_0}(\gamma_1), \tag{4.4}$$

where $\sigma_{\mathcal{F}_0}$ is defined on the extended up to the boundary broken bicharacteristics by continuity.

Thus we constructed FIO $\widetilde{\mathcal{F}}_0:\ \mathcal{E}'_{r,\delta}(\partial Q)\times D'_{m,\delta}(Q_0)\longmapsto\hat{\mathcal{D}}'_m(Q)$ such that for any $(g,u)\in\mathcal{E}'_{r,\delta}(\partial Q)\times D'_{m,\delta}(Q_0)$

$$\widetilde{\mathcal{F}}_0(g,u)\Big|_Q=\mathcal{F}_0(g,u). \tag{4.5}$$

§5. THE PRINCIPAL SYMBOLS OF FIO $\mathcal{F}_j$, $j\geq 1$

First we consider a linear bundle with connection and describe a process of integration of sections of this bundle along curves belonging to the base.

Let $\mathcal{M}$ be a smooth manifold, L — a linear bundle over $\mathcal{M}$, ∇ — a linear connection on L, v — a section of L and $\gamma\ :\ [s_0,s_1]\longrightarrow\mathcal{M}$ — a continuous curve on $\mathcal{M}$. We define the integral of v along γ:

$$\int v(s)\,ds=\int_{s_0}^{s_1}f(\gamma(s))\,\omega(\gamma(s_1))\,ds, \tag{5.1}$$

where ω is a horizontal with respect to ∇ non-vanishing section of L over γ, f is a function defined on γ by the relation $v=f\omega$. Notice that the integral in (5.1) does not depend on the choice of horizontal section ω. For, if ω_1 is another horizontal non-vanishing section such that $\omega=\rho\omega_1$ and ξ is a vector field $d\gamma\left(\frac{\partial}{\partial s}\right)$, then

$$0=\nabla_\xi\omega=\nabla_\xi(\rho\omega_1)=(\xi\rho)\omega_1+\rho\nabla_\xi\,\omega_1=(\xi\rho)\omega_1,$$

and $\rho = \text{const}$ along γ. So $v = f_1\omega_1$ implies $f_1 = \rho f$ and

$$\int_{s_0}^{s_1} f(\gamma(s))\,\omega(\gamma(s_1))\,ds = \int_{s_0}^{s_1} f_1(\gamma(s))\,\omega_1(\gamma(s_1))\,ds.$$

Denote by $\mathcal{F}_b$ the FIO with canonical relation

$$C_b = \{(z_1, z_2) \in \mathring{T}^*\partial Q \times \mathring{T}^*Q| \quad \pi_Q z_2 \in \partial Q,\ \pi_b z_2 = z_1\} \tag{5.2}$$

and the symbol $B(\pi_Q z_1) \bigotimes \nu_{\partial Q} \bigotimes |dx|^{1/2}$ ([14]). Notice that

$$\mathcal{F}_b \in I^{1/4}(\partial Q, \widetilde{Q}, C_b, \Omega^{1/2}_{\partial Q \times \widetilde{Q}} \times \mathrm{Hom}(E^r(\partial Q), E^m(\widetilde{Q})).$$

We are going to construct a sequence of FIO $\widetilde{\mathcal{F}}_j$, $j = 1, 2, ...$, with the same canonical relation $\widetilde{C}$ and with the principal symbols defined by the recursion relations and satisfying the conditions

$$\widetilde{\mathcal{F}}_j \in I^{\frac{-1}{4}-j}(\widetilde{Q}, \partial Q, \widetilde{C}, \Omega^{1/2}_{\widetilde{Q}\times\partial Q} \bigotimes \mathrm{Hom}(E^m(Q), E^r(\partial Q))), \tag{5.3}$$

$$A \circ \sum_{j=0}^{l} \widetilde{\mathcal{F}}_j \in I^{\frac{-1}{4}-l}(\widetilde{Q}, \partial Q, \widetilde{C}, \Omega^{1/2}_{\widetilde{Q}\times\partial Q} \bigotimes \mathrm{Hom}(E^m(\widetilde{Q}), E^r(\partial\widetilde{Q}))), \tag{5.4}$$

$$\mathcal{F}_b \circ \sum_{j=0}^{l} \widetilde{\mathcal{F}}_j - Id \in I^{-1-l}(\partial Q, \partial Q, C_b \circ \widetilde{C}, \Omega^{1/2}_{\partial Q\times\partial Q} \bigotimes \mathrm{Hom}(E^r(\widetilde{Q}), E^r(\partial\widetilde{Q}))), \tag{5.5}$$

One can write (5.4) as a condition on symbols:

$$\sigma\left(A \circ \sum_{j=0}^{l} \widetilde{\mathcal{F}}_j\right) \in S^{\frac{n}{2}-1-l}(\widetilde{C}, \Omega^{1/2}_{\widetilde{C}} \bigotimes \mathrm{Hom}(E^m(\widetilde{C}), E^r(\widetilde{C})). \tag{5.6}$$

Also the following condition should be satisfied

$$P\sigma\left(A \circ \sum_{j=0}^{l} \widetilde{\mathcal{F}}_j\right) \in S^{\frac{n}{2}-1-l}(\widetilde{C}, \Omega^{1/2}_{\widetilde{C}} \bigotimes \mathrm{Hom}(E^m(\widetilde{C}), E^r(\widetilde{C})). \tag{5.7}$$

It should be noted that the composition of FIO $\mathcal{F}_b$ and $\widetilde{\mathcal{F}}_j$ is correctly defined, because Condition 4 guarantees that the composition of canonical relations C_b and $\widetilde{C}$ is transversal (§7). Also, FIO $\widetilde{\mathcal{F}}_0$ constructed in §§2,3 satisfies to all of these conditions for $l = 0$ (it will be shown in §7).

Let $z_0 \in \mathrm{Char}A$, $M_1(z)$ be a smooth $(m-1) \times m$–matrix valued function, the columns of which are basis vectors of the vector space $(I - P(z))\mathcal{R}^m$. $M_1(z)$ is defined up to right action of the group $GL(m-1, \mathcal{R})$. It follows from Condition 1 that $\sigma_A(z)$ is an automorphism of the vector space $(I - P(z))\,\mathcal{R}^m$.
Consider smooth $m \times m$–matrix valued function

$$M(z) = M_1(z) \left[M_1^{\mathrm{tr}}(z)\, \sigma_A(z)\, M_1(z) \right]^{-1} M_1^{\mathrm{tr}}(z) \tag{5.8}$$

in a neighborhood of z_0. Invariance of $M(z)$ with respect to right action of $GL(m-1, \mathcal{R})$ on $\widetilde{M_1}(z)$ implies that $M(z)$ does not depend on the choice of a basis in $(I - P(z))\mathcal{R}^m$ and so $M(z)$ is corectly defined on $\widetilde{\mathrm{Char}}A$.
Denote by $M_C(\gamma)$, $\widetilde{K}_C$ and $\widetilde{\nabla}_C$ a smooth matrix valued function, linear bundle and linear connection obtained by pull back of $M(z)$, $\widetilde{K}_C$ and ∇ with respect to the mapping $\widetilde{\chi} : \widetilde{C} \longrightarrow \widetilde{\mathrm{Char}}A$.

Let us consider the vector bundle $\mathrm{Hom}(E^r(\widetilde{\mathrm{Char}}A), E^m(\widetilde{\mathrm{Char}}A))$ over $\widetilde{\mathrm{Char}}A$. Its fiber is the vector space of $m \times r$–matrices. Let $\mathcal{M}$ be the vector bundle obtained by pull back of $\mathrm{Hom}(E^r(\widetilde{\mathrm{Char}}A), E^m(\widetilde{\mathrm{Char}}A))$ with respect to $\chi : \widetilde{C} \longrightarrow \widetilde{\mathrm{Char}}A$. Denote by $\mathcal{M}_\Omega$ the tensor product $\mathcal{M} \bigotimes \Omega^{1/2}$ over $\widetilde{C}$. The principal symbols of the operators $\widetilde{\mathcal{F}}_j$ will be the sections of $\mathcal{M}_\Omega$. Let us consider the following smooth sections

$$f_j(\gamma) = M(\gamma)\,\sigma\left(A \circ \sum_{i=0}^{j-1} \widetilde{\mathcal{F}}_j\right) \tag{5.9}$$

of $\mathcal{M}_\Omega$ and FIO $\widetilde{\mathcal{F}}_{j,1}$ with the canonical relation $\widetilde{C}$ and the principal symbols f_j. From (5.8) we have

$$P\sigma(A \circ \mathcal{F}_{j,1}) \in S^{\frac{n}{2}-1-j}(\widetilde{C}, \Omega^{1/2}_{\widetilde{C}} \bigotimes \mathrm{Hom}(E^m(\widetilde{C}), E^r(\widetilde{C})). \tag{5.10}$$

Now we define the process of pull back of a broken bicharacteristic $\gamma \in \widetilde{C}$ from $\mathrm{Char}A$ to $\widetilde{C}$ in the following way:

with any broken bicharacteristic $\gamma \in \widetilde{C}$, $\gamma\,:\,[s_0, s_1] \setminus J \longrightarrow \mathrm{Char}A$, we associate piecewise continuous function $\gamma \in \widetilde{C}$, $\gamma\,:\,[s_0, s_1] \setminus J \longrightarrow \widetilde{C}$, such that for any $\widetilde{s} \in [s_0, s_1] \setminus J$, $\gamma(\widetilde{s})|\,[s_0, \widetilde{s}] \setminus J \longmapsto \mathrm{Char}A$ and $\widetilde{\gamma}(\widetilde{s}) = \gamma|_{[s_0,s_1]\setminus J}$. Consider the smooth section

$$\sigma^j = -P\sigma\left(A\widetilde{\mathcal{F}}_{j,1} + \sum_{i=0}^{j-1} A\widetilde{\mathcal{F}}_i\right) \tag{5.11}$$

of $\mathcal{M}_\Omega$. Let $\gamma \in \widetilde{C}$ and $\widetilde{\gamma}$ is a corresponding broken line on $\widetilde{C}$.

For any points γ_1 and γ_2 belonging to $\widetilde{\gamma}$ denote by $\mathcal{R}^C_{\gamma_{1,2}}$ the operator of the parallel transfer along the bicharacteristic curve $\gamma_{1,2}$ connecting $\widetilde{\gamma}_1 = \widetilde{\gamma}(\widetilde{s}_1)$ and $\widetilde{\gamma}_2 = \widetilde{\gamma}(\widetilde{s}_2)$. Denote by $\mathbf{S}^{C,j}_{\widetilde{\gamma}_1,\widetilde{\gamma}_2}$ the transfer operator in $\mathcal{M}_\Omega$ generated by integration of the section σ^j along $\widetilde{\gamma}_{1,2}$, so that if v belongs to the fiber of $\mathcal{M}_\Omega$ over the point $\widetilde{\gamma}_1$ then

$$\mathbf{S}^{C,j}_{\widetilde{\gamma}_1,\widetilde{\gamma}_2} v = \mathcal{R}^C_{\widetilde{\gamma}_1,\widetilde{\gamma}_2} v + \int_{\widetilde{\gamma}_{1,2}} \exp \mu_C(\widetilde{\gamma}(s))\,\sigma^j(\widetilde{\gamma}(s))\,ds \tag{5.12}$$

belongs to the fiber over the point $\widetilde{\gamma}$.

Let $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ be the projections of $\mathring{T}^*\widetilde{Q} \times \mathring{T}^*\partial Q$ to the first and second cofactors. Consider the composition

$$C_b \circ \widetilde{C} = \{(z_1, z_2, \gamma)| \quad (z_1, z_2) \in C_b,\, \gamma \in \partial C,\, \widetilde{\pi}_1 \circ \tau(\gamma) = z_2\} \tag{5.13}$$

of the canonical relations C_b and $\widetilde{C}$ and the mapping $\imath_b : C_b \circ \widetilde{C} \longrightarrow \mathring{T}^*\partial Q \times \mathring{T}^*\partial Q$ such that $\imath_b(z_1, z_2, \gamma) = (z_1, \widetilde{\pi}_2(\tau(\gamma)))$. Then $(C_b \circ \widetilde{C}, \imath_b)$ is a homogeneous immersed Lagrangian submanifold in $\mathring{T}^*\partial Q \times \mathring{T}^*\partial Q$. We define the mapping $\varphi|\ \partial C \longmapsto C_b \circ \widetilde{C}$:

$$\varphi(\gamma) = ((\pi_b \circ \widetilde{\pi}_1 \circ \tau)(\gamma), (\pi_1 \circ \tau)(\gamma), \gamma). \tag{5.14}$$

One can see that φ is diffeomorphism, $\imath_b \circ \varphi = (\pi_b \circ \beta_1) \times (\pi_b \circ \kappa_0)$. Thus $(\partial C, \imath_b \circ \varphi)$ is a homogeneous immersed Lagrangian submanifold in $\mathring{T}^*\partial Q \times \mathring{T}^*\partial Q$ and there exists a diffeomorphism from $(\partial C, \imath_b \circ \varphi)$ to $(C_b \circ \widetilde{C}, \imath_b)$.

Denote by $\mathcal{M}_\Omega(\partial C)$ the restriction of $\mathcal{M}_\Omega$ to $\partial C \subset \widetilde{C}$. Now we consider the following smooth section of $\mathcal{M}_\Omega(\partial C)$

$$\sigma_b^j(\gamma) = \Omega_0(\chi(\gamma))\, \sigma_{Id - \mathcal{F}_b \circ \sum\limits_{i=0}^{j-1} \widetilde{\mathcal{F}}_i}(\gamma), \tag{5.15}$$

where $\gamma \in \partial C$. If $\gamma \in \widetilde{C}$, $J_\gamma = (s_0, s_1, ..., s_k)$, and $\widetilde{\gamma}$ is the associated broken line on $\widetilde{C}$, then $\widetilde{\gamma}(s_1), ..., \widetilde{\gamma}(s_k) \in \widetilde{C} \sim \partial_- C \subset \partial C$, $\kappa_0(\gamma) \in \mathrm{Char}_b^+ A \sim \partial_0 C \subset \partial C$.

Let us define the principal symbol of $\mathcal{F}_{j,2}$:

$$\begin{aligned} \sigma_{\mathcal{F}_{j,2}}(\gamma) = \exp\left[-\mu_C(\gamma)\right]\, \mathcal{S}^{C,j}_{\widetilde{\gamma}(s_k),\gamma} \Big[\sigma_b^j(\widetilde{\gamma}(s_k)) + \\ + \Omega(\kappa_k(\gamma))\, \mathcal{S}^{C,j}_{\widetilde{\gamma}(s_{k-1}),\widetilde{\gamma}(s_k)} \Big[\sigma_b^j(\widetilde{\gamma}(s_{k-1})) + \Omega(\kappa_{k-1}(\gamma))\, \mathcal{S}^{C,j}_{\widetilde{\gamma}(s_{k-2}),\widetilde{\gamma}(s_{k-1})} \cdots \\ \ldots \Big[\sigma_b^j(\widetilde{\gamma}(s_1)) + \Omega(\kappa_1(\gamma))\, \mathcal{S}^{C,j}_{\widetilde{\gamma}(s_1),\widetilde{\gamma}(s_0)} \sigma_b^j(\kappa_0(\gamma))\Big] \ldots \Big]. \end{aligned} \tag{5.16}$$

Then we determine FIO $\widetilde{\mathcal{F}}_j$ to be

$$\widetilde{\mathcal{F}}_j = \widetilde{\mathcal{F}}_{j,1} + \widetilde{\mathcal{F}}_{j,2}.$$

Notice that if $\widetilde{\mathcal{F}}_j$ is FIO with the canonical relation C and the principal symbol $\sigma_{\mathcal{F}_j} = \sigma_{\widetilde{\mathcal{F}}_j}\Big|_C$, then for any $g \in \mathcal{E}'(\partial Q)$ one can obtain

$$\mathcal{F}_j g = \widetilde{\mathcal{F}}_j g\Big|_Q . \tag{5.17}$$

§6. MAIN RESULT

Let us consider the asymptotic sum

$$\mathcal{F} \sim \sum_{j=0}^{\infty} \mathcal{F}_j \tag{6.1}$$

and a function $\rho_\delta(x) \in C^\infty(\mathbf{R}^{n+1})$ such that $\rho_\delta(x) = 0$ in $\delta/2$–neighbourhood of $\partial Q \cap Q_0$ and $\rho_\delta(x) = 1$ outside of δ–neighbourhood of $\partial Q \cap Q_0$.

We define the operator

$$E^\delta \ : \quad \mathcal{E}'_{r,\delta}(\partial Q) \times D'_{m,\delta}(Q_0) \longrightarrow \hat{\mathcal{D}}'_m(Q) \tag{6.2}$$

such that

$$E^\delta(g, u_0) \bigotimes |dx|^{1/2} = \rho_\delta(x) \times \mathcal{F}(g \bigotimes \nu_{\partial Q}, u_0 \bigotimes |dx'|^{1/2}), \tag{6.3}$$

where $\nu_{\partial Q}$ is non–vanishing semidensity on ∂Q, defined in §3.

In order to formulate the theorem we need the following

Condition 6. For any $x \in \overline{Q}$ the set $\chi^{-1}(x)$ is finite.

Theorem 6.1 If Conditions 1–6 hold, then the operator E_δ is the parametrix of the Problem (1.1) — (1.2).

§7. PROOF OF THEOREM 6.1

Let X and Y be C^∞–manifolds, $E \longrightarrow X$, $F \longrightarrow Y$ — vector bundles and $(C, \imath)$ be canonical relation from Y to X. We call $(C, \imath)$ a local–finite canonical relation if for any $z \in \imath(C)$ there exists a neighbourhood $U_z \subset \overset{\circ}{T}{}^*X \times \overset{\circ}{T}{}^*Y$ and a positive integer k_z such that the restriction of $\imath$ to $\imath^{-1}(U_z)$ is k_z–multiple covering.

Let $(C_1, \imath_1)$ and $(C_2, \imath_2)$ be locally–finite canonical relations from Y to X. We say that $(C_1, \imath_1)$ is reduced to $(C_2, \imath_2)$ if there exists a smooth mapping f from C_1 to C_2 such that $\imath_1 = \imath_2 \circ f$. Let $\mathcal{F}_i \in I^m(X \times Y, C_i)$, $i = 1, 2$, and the mapping f reduces $(C_1, \imath_1)$ to $(C_2, \imath_2)$. Denote by $\widetilde{\mathcal{F}}_1$ the FIO with canonical relation $(C_2, \imath_2)$ and the principal symbol

$$\sigma_{\widetilde{\mathcal{F}}_1}(\gamma) = \sum \sigma_{\mathcal{F}_1}(\gamma_i), \tag{7.1}$$

where the summation is taken over all preimages $\gamma_i \in C_1$ of the point $\gamma \in C_2$ with respect to f. Then

$$\widetilde{\mathcal{F}}_1 \in I^m(X \times Y, C_2)$$

and

$$\widetilde{\mathcal{F}}_1 = \mathcal{F}_2(\mathrm{mod} I^{m-1}(X \times Y, C_2)). \tag{7.2}$$

Let X, Y, Z be C^∞–manifolds, $E \longrightarrow X$, $F \longrightarrow Y$, $G \longrightarrow Z$ — vector bundles, $(C_i, \imath_i)$, $i = 1, 2$ — homogeneous immersed Lagrangian submanifolds of the symplectic manifolds $(\overset{\circ}{T}{}^*X \times \overset{\circ}{T}{}^*Y, \omega_1)$ and $(\overset{\circ}{T}{}^*Y \times \overset{\circ}{T}{}^*Z, \omega_2)$, where $\omega_1 = \omega_{T^*X} - \omega_{T^*Y}$, $\omega_2 = \omega_{T^*Y} - \omega_{T^*Z}$. Then $(C_1 \times C_2, \imath_1 \times \imath_2)$ is the immersed submanifold in $\overset{\circ}{T}{}^*X \times \overset{\circ}{T}{}^*Y \times \overset{\circ}{T}{}^*Y \times \overset{\circ}{T}{}^*Z$. Denote by Δ the diagonal in $\overset{\circ}{T}{}^*Y \times \overset{\circ}{T}{}^*Y$

and by C the preimage of $\overset{\circ}{T}{}^*X \times \Delta \times \overset{\circ}{T}{}^*Z$ with respect to the mapping $\imath_1 \times \imath_2$. We assume that $\imath_1 \times \imath_2$ is transversal to $\overset{\circ}{T}{}^*X \times \Delta \times \overset{\circ}{T}{}^*Z$. Then C is the submanifold in $C_1 \times C_2$ (see [15]).

Let

$$\pi_{X\times Z} \ : \quad T^*X \times T^*Y \times T^*Y \times T^*Z \longrightarrow T^*X \times T^*Z$$

be the canonical projection and

$$\imath = \pi_{X\times Z} \circ (\imath_1 \times \imath_2)|_C \ : \quad C \longrightarrow T^*X \times T^*Z. \tag{7.3}$$

The homogeneous immersed Lagrangian submanifold $(C, \imath)$ in $(\overset{\circ}{T}{}^*X \times \overset{\circ}{T}{}^*Z, \omega_{T^*X} - \omega_{T^*Z})$ we call a transversal composition of canonical relations $(C_1, \imath_1)$ and $(C_2, \imath_2)$. Notice that if $(C_1, \imath_1)$ and $(C_2, \imath_2)$ are locally–finite then their transversal composition is locally–finite too.

Now we can pass to the proof of the theorem. Let us consider the immersed Lagrangian submanifold $(T^*\partial Q \cup \hat{C}, \hat{\imath})$ in $T^*\partial Q \times T^*\partial Q$, where

$$\hat{\imath}|_{T^*\partial Q} = \mathrm{id} \times \mathrm{id}, \quad \hat{\imath}|_{\hat{C}} = \chi \times (\pi_b \circ \chi_0).$$

Define the mapping $f \ : \partial C \longrightarrow T^*\partial Q \cup \hat{C}$ in the following way: $f|_{\partial_0 C} = \pi_b$ is r–multiple covering, $f|_{\partial_- C} \ : \partial_- C \longrightarrow \hat{C}$ is diffeomorphism, $f|_{\partial_+ C} \ : \ \partial_+ C \longrightarrow \partial_- C \sim \hat{C}$ is r–multiple covering. The following relation holds

$$\hat{\imath} \circ f = \imath_b \circ \varphi. \tag{7.4}$$

We consider FIO $\mathcal{F}_b \circ \widetilde{\mathcal{F}}_j$ with canonical relation $C_b \circ \widetilde{C}$. The mapping f reduces

$C_b \circ \widetilde{C}$ to $(T^*\partial Q \cup \hat{C}, \hat{\imath})$. From (7.4) and (7.1) we have

$$\mathcal{F}_b \circ \widetilde{\mathcal{F}}_j = \hat{\mathcal{F}}_j (\mathrm{mod} I^{j-1}(\partial Q, \partial Q, T^*\partial Q \cup \hat{C}, \Omega^{1/2}_{\partial Q \times \partial Q} \bigotimes \mathrm{Hom}(E^r(\partial Q), E^r(\partial Q)))), \tag{7.5}$$

where $\hat{\mathcal{F}}_j$ is FIO with canonical relation $(T^*\partial Q \cup \hat{C}, \hat{\imath})$ and the principal symbol:

a) for $\gamma \in T^*\partial Q$

$$\sigma_{\hat{\mathcal{F}}_j}(\gamma) = \sum_{j=1}^{r} \sigma_{\mathcal{F}_b \circ \widetilde{\mathcal{F}}_j}(\widetilde{\gamma}_j), \tag{7.6}$$

b) for $\gamma \in \hat{C}$

$$\sigma_{\hat{\mathcal{F}}_j}(\gamma) = \sigma_{\mathcal{F}_b \circ \widetilde{\mathcal{F}}_j}(\gamma) + \sum_{j=1}^{r} \sigma_{\mathcal{F}_b \circ \widetilde{\mathcal{F}}_j}(\widetilde{\gamma}_j), \tag{7.7}$$

where $(\widetilde{\gamma}_1, ..., \widetilde{\gamma}_r) = f^{-1}(\gamma)$.

In order to prove (5.5) for $l = 0$ let us write the principal symbol of the operator $\mathcal{F}_b \circ \widetilde{\mathcal{F}}_0$. Since $\varphi|\ \partial C \mapsto C_b \circ \widetilde{C}$ is a diffeomorphism, we have

$$\varphi^* \sigma_{\mathcal{F}_b \circ \widetilde{\mathcal{F}}_0} \in S^{n/2}(\partial C, \Omega^{1/2}_{\partial C} \bigotimes \mathrm{Hom}(E^r(\partial C), E^r(\partial C))).$$

It follows from (3.13), (4.3), (4.4) and from formula of composition for FIO ([12]), that

a) for $\gamma \in \mathrm{Char}^+_b A \sim \partial_0 C$

$$\Phi^* \sigma_{\mathcal{F}_b \circ \widetilde{\mathcal{F}}_0} = B(\pi_Q \circ \pi_b(\gamma)) \Omega_0(\pi_b(\gamma)) \bigotimes \pi^*_b \nu_{\partial Q} \bigotimes \pi^*_b \nu_{\partial Q}, \tag{7.8}$$

b) for $\gamma = (\gamma_1, \gamma_2) \in \partial_+ C \cup \partial_- C$

$$\varphi^* \sigma_{\mathcal{F} \circ \widetilde{\mathcal{F}}} = \exp[-\mu_c(\gamma_1)]\, \pi^*_Q \nu_{\partial Q} \bigotimes B(\pi_Q \circ \pi_b(\gamma_2)) \pi^*_Q |dx|^{-1/2} \times R^{\nabla}_{\kappa_{k(\gamma_1)-1}(\gamma_1), \kappa(\gamma_1)} \Omega(\kappa_{k(\gamma_1)-1}(\gamma_1) \circ ...$$

$$... \circ R^{\nabla}_{\kappa_0(\gamma_1), \kappa_1(\gamma_1)} (\Omega_0(\kappa_0(\gamma_1) \bigotimes \pi^*_Q |dx|^{1/2}) \bigotimes \pi^*_Q \nu_{\partial Q}. \tag{7.9}$$

From (7.6) – (7.9) we obtain

a) for $\gamma \in T^*\partial Q$

$$\sigma_{\hat{\mathcal{F}}_0}(\gamma) = \sum_{j=1}^{r} B(\pi_Q \circ \pi_b(\widetilde{\gamma}_j))\Omega_0(\pi_b(\widetilde{\gamma}_j)) \bigotimes \pi_b^* \nu_{\partial Q} \bigotimes \pi_b^* \nu_{\partial Q}. \tag{7.10}$$

From this and from (3.10), (3.11)

$$\sigma_{\hat{\mathcal{F}}_0}(\gamma) = B(\pi_Q(\gamma)) \sum_{j=1}^{r} P(\widetilde{\gamma}_j)V(\gamma)[B(\gamma)V(\gamma)]^{-1} \bigotimes \pi_b^* \nu_{\partial Q} \bigotimes \pi_b^* \nu_{\partial Q}. \tag{7.11}$$

Let $E_+(\gamma)$ be a linear subspace in $\mathbf{R}^m$ spanned on the columns of the matrix $V(\gamma)$. The projector $\sum\limits_{j=1}^{r} P(\widetilde{\gamma}_j)$ acts on $E_+(\gamma)$ as an identity operator, so we obtain

$$\sigma_{\hat{\mathcal{F}}_0}(\gamma) = \mathrm{Id} \bigotimes \pi_b^* \nu_{\partial Q} \bigotimes \pi_b^* \nu_{\partial Q}. \tag{7.12}$$

b) For $\gamma \in \hat{C}$

$$\sigma_{\hat{\mathcal{F}}_0}(\gamma) = \exp[-\mu_C(\gamma)]\, \pi_Q^* \nu_{\partial Q} \bigotimes B(\pi_Q \circ \chi(\gamma))\pi_Q^* |dx|^{-1/2} \times R^{\nabla}_{\kappa_{k(\gamma)-1}(\gamma),\chi(\gamma)} \Omega(\kappa_{k(\gamma)-1}(\gamma) \circ ...$$

$$... \circ R^{\nabla}_{\kappa_0(\gamma),\kappa_1(\gamma)} (\Omega_0(\kappa_0(\gamma) \bigotimes \pi_Q^* |dx|^{1/2}) \bigotimes \pi_Q^* \nu_{\partial Q} +$$

$$+ \sum_{j=1}^{r} \exp[-\mu_c(\widetilde{\gamma}_j)]\, \pi_Q^* \nu_{\partial Q} \bigotimes B(\pi_Q \circ \chi(\widetilde{\gamma}_j)) \bigotimes \pi_Q^* |dx|^{-1/2} \times \Omega(\kappa(\widetilde{\gamma}_j)) R^{\nabla}_{\kappa_{k(\widetilde{\gamma}_j)-1}(\widetilde{\gamma}_j),\chi(\widetilde{\gamma}_j)} \Omega(\kappa_{k(\widetilde{\gamma}_j)-1}(\widetilde{\gamma}_j) \circ ...$$

$$... \circ R^{\nabla}_{\kappa_0(\widetilde{\gamma}_j),\kappa_1(\widetilde{\gamma}_j)} (\Omega_0(\kappa_0(\widetilde{\gamma}_j) \bigotimes \pi_Q^* |dx|^{1/2}) \bigotimes \pi_Q^* \nu_{\partial Q},$$

where $(\widetilde{\gamma}_1, ..., \widetilde{\gamma}_r) = f^{-1}(\gamma)$.

Notice that

$$\pi_Q \circ \chi(\widetilde{\gamma}_j) = \pi_Q \circ \chi(\gamma),\ \kappa_i(\widetilde{\gamma}_j) = \kappa_i(\gamma),$$

$$R^{\nabla}_{\kappa_i(\widetilde{\gamma}_j),\kappa_{i+1}(\widetilde{\gamma}_j)} = R^{\nabla}_{\kappa_i(\gamma),\kappa_{i+1}(\gamma)},\ i = 0, ..., k_{\gamma-1},\ \mu_C(\widetilde{\gamma}_j) = \mu_C(\gamma),\ j = 1, ..., r.$$

So we have

$$\sigma_{\hat{\mathcal{F}}_0}(\gamma) = \exp[-\mu_c(\gamma)]\, \pi_Q^* \nu_{\partial Q} \bigotimes B(\pi_Q \circ \chi(\gamma))\pi_Q^* |dx|^{-1/2} \times$$

$$(I+\sum_{j=1}^{r}\Omega(\chi(\widetilde{\gamma}_j)))R^{\nabla}_{\kappa_{k(\gamma)-1}(\gamma),\chi(\gamma)}\Omega(\kappa_{k(\gamma)-1}(\gamma)\circ ...$$

$$...\circ R^{\nabla}_{\kappa_0(\gamma),\kappa_1(\gamma)}(\Omega_0(\kappa_0(\gamma)\bigotimes \pi_Q^*|dx|^{1/2})\bigotimes \pi_Q^*\nu_{\partial Q}. \qquad (7.13)$$

From (3.12) we obtain

$$B(\pi_Q\circ\chi(\gamma))(I+\sum_{j=1}^{r}\Omega(\chi(\widetilde{\gamma}_j))) = B(\pi_Q\circ\chi(\gamma))-\sum_{j=1}^{r}P(\chi(\widetilde{\gamma}_j))\times W(\pi_b\circ\chi(\widetilde{\gamma}_j))B(\pi_Q\circ\chi(\widetilde{\gamma}_j)) =$$

$$= B(\pi_Q\circ\chi(\widetilde{\gamma}_j)) - B(\pi_Q\circ\chi(\gamma))\times\left(\sum_{j=1}^{r}P(\chi(\widetilde{\gamma}_j))\right)\times W(\pi_b\circ\chi(\gamma))B(\pi_Q\circ\chi(\gamma)). \qquad (7.14)$$

The projector $\sum\limits_{j=1}^{r}P(\chi(\widetilde{\gamma}_j))$ acts on $E_+(\chi(\gamma))$ as the identity operator, so by (3.10) we can write

$$B(\pi_Q\circ\chi(\gamma))\times\left(\sum_{j=1}^{r}P(\chi(\widetilde{\gamma}_j))\right)\times W(\pi_b\circ\chi(\gamma))B(\pi_Q\circ\chi(\gamma)) = B(\pi_Q\circ\chi(\gamma))\times$$

$$\times\left(\sum_{j=1}^{r}P(\chi(\widetilde{\gamma}_j))\right)V(\pi_b\circ\chi(\gamma))[B(\pi_Q\circ\chi(\gamma))\times V(\pi_b\circ\chi(\gamma))]^{-1}B(\pi_Q\circ\chi(\gamma)) = B(\pi_Q\circ\chi(\gamma)). \qquad (7.15)$$

It follows from (7.13) – (7.15) that $\sigma_{\hat{\mathcal{F}}_0}(\gamma)=0$ for $\gamma\in\hat{C}$, so (5.5) is proved for $l=0$. The relation (5.5) for $l=1,2,...$ are proved similarly.

The proof of (5.6) is standard and we turn to the relations (5.7).

Let us show that

$$P\sigma_{-1}(A\mathcal{F}_0)=0. \qquad (7.16)$$

It is well known (see [12], [16]) that

$$\sigma_{-1}(A\mathcal{F}_0)=\{\sigma_A,\sigma_{\mathcal{F}_0}\}+\sigma_{-1}(A)\sigma_{\mathcal{F}_0}.$$

Since

$$P\sigma_{-1}(A\mathcal{F}_0) = P(\{\sigma_A - \lambda I, \sigma_{\mathcal{F}_0}\} + \sigma_{-1}(A)\sigma_{\mathcal{F}_0}) + P\{\lambda, \sigma_{\mathcal{F}_0}\} = \mu\sigma_{\mathcal{F}_0} + P\mathcal{D}_{H_p}\sigma_{\mathcal{F}_0} =$$

$$= \mu\sigma_{\mathcal{F}_0} + \bigtriangledown_{H_p}\sigma_{\mathcal{F}_0} = (\bigtriangledown_{H_p} + \mu)\exp[-\mu_C(\gamma)] \times \chi^* R^{\bigtriangledown}_{\kappa_{k(\gamma)-1}(\gamma),\chi(\gamma)} v = \bigtriangledown_{H_p} R^{\bigtriangledown} v = 0.$$

In the same way

$$P\sigma(A \circ \mathcal{F}_j) = P\sigma(A \circ \mathcal{F}_{j,1}) + P\sigma(A \circ \mathcal{F}_{j,2}) = P\sigma(A \circ \mathcal{F}_{j,1}) + (\bigtriangledown_{H_p} + \mu)\sigma(\mathcal{F}_{j,2}).$$

It follows from (5.16), (5.12) and (5.11) that the last expression is equal to

$$P\sigma(A \circ \mathcal{F}_{j,1}) - P\sigma(A \circ \widetilde{\mathcal{F}}_{j,1}) - P\sigma(\sum_{i=0}^{j-1} A \circ \mathcal{F}_j),$$

so that (5.7) holds.

It remains to show that the difference between $E_\delta(g, u_0)$ and the exact solution of the problem (1.1) – (1.2) belongs to $C^\infty(\overline{Q})$. Let $V_\delta(\partial Q)$ be δ–neighbourhood of ∂Q in $\mathbf{R}^{n+1}$. For $x \notin V_\delta(\partial Q)$ we have

$$\widetilde{\mathcal{F}}(g, u_0) - E^\delta(g, u_0) = (1 - \rho_\delta(x))\,\widetilde{\mathcal{F}}(g, u_0) = 0.$$

On the other hand, $\widetilde{\mathcal{F}}(g, u_0) \in C^\infty(\overline{Q} \subset V_\delta(\partial Q))$. Therefore $\omega = E_\delta(g, u_0) - u$ is the solution of the mixed problem with C^∞–data and these data are C^∞–compatible.

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