Newtonian iterative scheme with simultaneous iterations of inverse derivative

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A modification of the Continuous Analogy of Newton Method for the numerical solving of the nonlinear problems is suggested. It permits one to replace the inversion of the derivative operator on every step of iterations by its inversion only in the initial approximation point. Then the extended system of the differential equations in Hilbert space, introduced in the work, permits the realization of the iterative process with the simultaneous calculation of the inverse derivative operator. The convergence theorem is proved for the method under almost the same conditions as for CANM. Numerical calculations for the model problem (Kirchhoff equation) have shown the effectiveness and adequately fast convergence of the iterative schemes based on the suggested method.

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1 Introduction

Let us consider a nonlinear equation in Hilbert space

$$\varphi(x) = 0 \ . \tag{1}$$

A principal point of the realization of Continuous Analogy of Newton Method (CANM) ([1–3]) for solving (1) is the inversion of the operator $\varphi'(x)$. Such

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inversion is often difficult and could be a reason for a decreasing of the accuracy of calculations. In [4], the method of parameter variation has been suggested. It reduces the inversion of a matrix-function $A(\lambda)$ to the nonlinear matrix differential equation

$$\frac{\mathrm{dB}(\lambda)}{\mathrm{d}\lambda} = -B(\lambda)\frac{\mathrm{dA}(\lambda)}{\mathrm{d}\lambda}B(\lambda) \ . \tag{2}$$

A combination of CANM with the method of parameter variation has been realized in [5], where the Newtonian Iterative Scheme with simultaneous inversion of $\varphi'(x)$ has been constructed. The similar algorithm, but in a framework of usual Newton Method, has been earlier investigated in [6].

In this paper we suggest another method realizing CANM with a simultaneous solving of the operator differential equation for the inverse derivative. However, in contrast to the papers cited above, for the inversion of φ'^{-1} we use a linear nonhomogeneous equation. Such modification has made it possible to prove the convergence theorem under almost the same conditions on φ as in CANM ([1]).

The paper is organized in the following way. In Sect. 2 the covergence theorem for CANM is brought. In Sect. 3 the proposed method is described and the convergence theorem is formulated. The last one is proved in Sect. 4. And in Sect. 5, the numerical example illustrating the applications of the method is examined.

2 Continuous Analogy of Newton Method

Let H be a real or a complex Hilbert space and $L(H)$ – the space of linear operators in H. Let us consider the equation (1) for $\varphi : H \to H$.

Denote by x_0 some initial approximation to the solution of (1) and by $B(x_0, r)$ the ball: $||x - x_0|| < r$. We assume that the following condition on φ holds.

Condition A. *There exist some positive real numbers r, c and continuous operator-valued function $F(x)$ from $B(x_0, r||\varphi(x_0)||)$ to $L(H)$ such that*

1) there exist Frechét derivative $\varphi'(x)$ and Gateaux derivatives $\varphi''(x)$ and $F'(x)$, and they are locally bounded in $B(x_0, r||\varphi(x_0)||)$;

2) for every $x \in B(x_0, r||\varphi(x_0)||)$ the spectrum of the operator $\varphi'(x)F(x)$ lies in the half-plane $Re z \geq c$;

3) for any $x \in B(x_0, r||\varphi(x_0)||)$

$$||F(x)|| \leq rc \ . \tag{3}$$

Now let us consider the following Cauchy problem in H:

$$x'(t) = -F(x(t))\varphi(x(t)), \quad 0 \leq t < \infty, \quad x(0) = x_0 \ . \tag{4}$$

Theorem 1 *If Condition A holds then*
1) there exists a solution $x = x(t), \quad t \in [0, \infty)$ *of the problem (4) and* $x(t) \in B(x_0, r||\varphi(x_0)||)$ *for all* $t \in [0, +\infty)$*;*
2) there exists

$$\lim_{t \to +\infty} x(t) = x^* \tag{5}$$

and x^* *is the solution of the problem (1).*

Remark 2 *a) If* $F(x) = [\varphi'(x)]^{-1}$ *then we get CANM. In that case* $c = 1$ *and Theorem 1 turns into the convergence theorem for CANM([1]);*

b) if $F(x)$ *is an identity operator one obtain the method of rapidest slope;*

c) $F(x) = [\varphi'(x)]^*$ *corresponds to the gradient method.*

Proof([1]). From the Condition A we get the local existence of a solution of the problem (4). Then from (4)

$$\varphi'(x(t))x'(t) = -\varphi'(x(t))F(x(t))\varphi(x(t)) \ . \tag{6}$$

Let us denote

$$\lambda(t) = \varphi(x(t)) \ . \tag{7}$$

Then from (4) we obtain

$$\lambda'(t) = -\varphi'(x(t))F(x(t))\lambda(t), \quad \lambda(0) = \varphi(x_0) \ . \tag{8}$$

Therefore

$$\lambda(t) = \exp\left[-\int_0^t \varphi'(x(s))F(x(s))ds\right]\varphi(x_0) \ . \tag{9}$$

From Condition A we come to the following estimates

$$||\lambda(t)|| \leq ||\varphi(x_0)||e^{-ct} \tag{10}$$

and

$$||x(t) - x_0|| \leq || \int_0^t x'(s)ds|| \leq \int_0^t ||F(x)||\,||\lambda(s)||ds \leq \tag{11}$$

$$\leq \frac{||F|| \times ||\varphi(x_0)||}{c}(1 - e^{-ct}) \leq r||\varphi(x_0)|| \ . \tag{12}$$

So $x(t)$ doesn't abandon the ball $B(x_0, r||\varphi(x_0)||)$ fot $t \in [0, \infty)$.

Thus from (10) we get

$$\lim_{t\to\infty} \varphi(x(t)) = \lim_{t\to\infty} \lambda(t) = 0 \ , \tag{13}$$

so $\lim_{t\to\infty} x(t)$ is the solution of the problem (1). $\square$

3 CANM with Simultaneous Inversion of φ' and Convergence Theorem

Now our goal is to develop the Continuous Newton Method with Simultaneous Inversion of φ'.

We consider the following system:

$$\begin{cases} \varphi(x) = 0 \\ \varphi'(x)Y - E = 0 \ , \end{cases} \tag{14}$$

where $Y \in L(H)$ and E is an identity operator. Let Y_0 be an approximation to $\varphi'(x_0)^{-1}$. Denote

$$\begin{pmatrix} \lambda(t) \\ \Lambda(t) \end{pmatrix} = \begin{pmatrix} \varphi(x(t)) \\ \varphi'(x(t))Y(t) - E \end{pmatrix} \ .$$

and consider the Cauchy problem

$$\begin{cases} \lambda'(t) = -\varphi'(x(t))Y(t)\lambda(t) \ , \\ \Lambda'(t) = -\rho^2(\varphi'(x(t))(\varphi'(x(t)))^*\Lambda(t) + \Lambda(t)\varphi'(x(t))(\varphi'(x(t)))^*)+ \\ +(\varphi''(x(t))x'(t))Y(t) \ , \end{cases} \tag{15}$$

$$\lambda(0) = \varphi(x_0), \quad \Lambda(0) = \varphi'(x_0)Y_0 - E \ ,$$

where ρ is a positive real number, and $\varphi''(x)$ for fixed $x \in H$ is Gateaux derivative which is a linear operator from H to $L(H)$ such that

$$\varphi'(x+\xi) - \varphi'(x) = \varphi''(x)\xi + \eta, \text{ and } ||\eta||_{L(H)}||\xi||^{-1} \to 0 \text{ for } \xi \to 0.$$

Then, instead of the (4), we consider the problem

$$\begin{cases} x'(t) = -Y(t)\varphi(x(t)) \ , \\ Y'(t) = -\rho^2((\varphi'(x(t))^*\varphi'(x(t))Y(t) + Y(t)\varphi'(x(t))(\varphi'(x(t)))^*)+ \\ +2\rho^2(\varphi'(x(t)))^* \ , \end{cases} \tag{16}$$

$$x(0) = x_0, \quad Y(0) = Y_0 \ .$$

We consider now the following condition on φ.

Condition B. *There exist some positive real numbers r and ϵ such that*

1) there exist Fréchet derivative $\varphi'(x)$ and Gateaux derivative $\varphi''(x)$ in $B(x_0, r||\varphi(x_0)||)$ and

$$||\varphi''||_r = \sup_{x \in B(x_0, r||\varphi(x_0)||)} \sup_{\xi \in H, ||\xi||=1} ||(\varphi''(x)\xi||_{L(H)} < \infty \ ;$$

2) for any $x \in B(x_0, r||\varphi(x_0)||)$ the operator $(\varphi'^*(x))$ is invertible and

$$||\varphi'^{*-1}||_r = \sup_{x \in B(x_0, r||\varphi(x_0)||)} ||(\varphi'^*(x))^{-1}|| < \infty \ ;$$

3) the following inequality holds

$$0 < \frac{max\{||Y_0||, ||\varphi'^{*-1}||_r\}}{1 - max\{||\varphi'(x_0)Y_0 - E||, \epsilon\}} < r \ . \tag{17}$$

Let us denote

$$||\varphi||_r = \sup_{x \in B(x_0, r||\varphi(x_0)||)} ||\varphi(x)|| \ ,$$

$$\rho_0 = \max\{||Y_0||, ||\varphi'^{*-1}||_r\} \, ||\varphi'^{*-1}||_r \sqrt{\frac{||\varphi''||_r ||\varphi||_r}{2\epsilon}} \ . \tag{18}$$

The following theorem establishes the convergence of the method.

Theorem 3 *If Condition B holds, then for $\rho > \rho_0$ and for $t \in [0, +\infty)$*
1) there exists the solution $(x(t), Y(t))$ of the problem (16) and

$$x(t) \in B(x_0, r||\varphi(x_0)||) \ , \tag{19}$$

$$||\varphi'(x(t))Y(t) - E|| \leq \ max\{||\varphi'(x_0)Y_0 - E||, \epsilon\} \ ; \tag{20}$$

2) there exists

$$\lim_{t \to +\infty} x(t) = x^*$$

and x^* is the solution of the problem (1).

4 The proof of Theorem 3

First we prove some lemmas. Let us consider the Cauchy problem

$$Y'(t) + A^*(t)A(t)Y(t) + Y(t)A(t)A^*(t) - 2\rho A^*(t) = 0, \quad Y(0) = Y_0, \tag{21}$$

where $A(t)$ is a given continuous operator-valued function on $[0, +\infty)$.

Lemma 4 *Let the operator $A^*(t)$ be invertible for every $t \in [0, T]$, $T \leq +\infty$ and the following estimates be valid*

$$||Y_0|| \leq C_1 \ ,$$

$$\rho||(A^*(t))^{-1}|| \leq C_1, \quad for \quad t \in [0, T] \ . \tag{22}$$

Then there exists the unique solution of the problem (21) on $[0, T]$ and

$$||Y(t)|| \leq C_1 \quad for \quad t \in [0, T] \ . \tag{23}$$

PROOF. As it has done for Riccati matrix equations in [7] , we consider the auxiliary system

$$\begin{cases} U'(t) - A(t)A^*(t)U(t) = 0, \quad U(0) = E \ , \\ V'(t) - 2\rho A^*(t)U(t) + A^*(t)A(t)V(t) = 0, \quad V(0) = Y_0 \ . \end{cases} \tag{24}$$

The operator $U(t) = \exp \int\limits_0^t A(s)A^*(s)ds$ is invertible, and so

$$Y(t) = V(t)U^{-1}(t) \ . \tag{25}$$

Now we introduce an operator-valued function

$$\Psi(t) = V^*(t)V(t) - C_1^2 U^*(t)U(t) = U^*(t)(Y^*(t)Y(t) - C_1^2 E)U(t). \tag{26}$$

Denote by $(,)$ the scalar product in H, then for any $\xi \in H$

$$\begin{gathered} \frac{\mathrm{d}}{\mathrm{dt}}(\xi, \Psi(t)\xi) = \\ = (\xi, [2\rho U^*(t)A(t)V(t) - V^*(t)A^*(t)A(t)V(t) + 2\rho V^*(t)A^*(t)U(t) - \\ -V^*(t)A^*(t)A(t)V(t) - 2C_1^2 U^*(t)A(t)A^*(t)U(t)]\xi) = \\ = 4\rho \mathrm{Re}(U(t)\xi, A(t)V(t)\xi) - 2||A(t)V(t)\xi||^2 - 2C_1^2||A^*(t)U(t)\xi||^2. \end{gathered}$$

From the inequalities

$$2\rho(U(t)\xi, A(t)V(t)\xi) \leq \rho^2||U(t)\xi||^2 + ||A(t)V(t)\xi||^2,$$

$$||U(t)\xi|| = ||(A^*(t))^{-1}A^*(t)U(t)\xi|| \leq ||(A^*(t))^{-1}|| \, ||A^*(t)U(t)\xi||$$

and from (22) we get

$$\frac{\mathrm{d}}{\mathrm{dt}}(\xi, \Psi(t)\xi) \leq 0 \ .$$

So

$$(\xi, \Psi(t)\xi) \leq (\xi, \Psi(0)\xi)$$

and

$$(\xi, U^*(t)(Y^*(t)Y(t) - C_1^2 E)U(t)\xi) \leq (\xi, U^*(0)(Y_0^* Y_0 - C_1^2 E)U(0)\xi) \leq 0.$$

Therefore

$$||Y(t)U(t)\xi||^2 - C_1^2||Y_0||^2||U(t)\xi||^2 \le ||Y_0\xi||^2 - C_1^2||\xi||^2 \le 0 \ .$$

The estimate (23) follows from the last inequality and from the invertibility of $U(t)$. $\square$

Now we consider the Cauchy problem

$$\Lambda'(t) + A(t)A^*(t)\Lambda(t) + \Lambda(t)A(t)A^*(t) - D(t) = 0, \quad \Lambda(0) = \Lambda_0, \quad (27)$$

where $\Lambda(t), A(t), D(t)$ are the given continuous operator-valued functions on $[0, +\infty]$.

Lemma 5 *Let an operator $A^*(t)$ be invertible for every $t \in [0, T]$, $T \le +\infty$ and the following estimates be valid*

$$||\Lambda_0|| \le C_2 \ ,$$

$$||D(t)||\,||(A^*(t))^{-1}||^2 \le 2C_2, \quad for \quad t \in [0, T] \ . \quad (28)$$

Then there exists a unique solution of the problem (27) on $[0, T]$ and

$$||\Lambda(t)|| \le C_2 \quad for \quad t \in [0, T] \ . \quad (29)$$

PROOF. The proof is very similar to the previous one. We consider

$$\begin{cases} U'(t) - A(t)A^*(t)U(t) = 0, \quad U(0) = E \ , \\ V'(t) - D(t)U(t) + A(t)A^*(t)V(t) = 0, \quad V(0) = \Lambda_0 \ . \end{cases} \quad (30)$$

Then

$$\Lambda(t) = V(t)U^{-1}(t) \quad (31)$$

and

$$\Psi(t) = V^*(t)V(t) - C_2^2U^*(t)U(t) = U^*(t)(\Lambda^*(t)\Lambda(t) - C_2^2E)U(t). \quad (32)$$

For any $\xi \in H$

$$\frac{\mathrm{d}}{\mathrm{dt}}(\xi, \Psi(t)\xi) = (\xi, [U^*(t)D^*(t)V(t) - 2V^*(t)A(t)A^*(t)V(t) +$$
$$+V^*(t)D(t)U(t) - 2C_2^2U^*(t)A(t)A^*(t)U(t)]\xi) =$$
$$= 2\mathrm{Re}(D(t)U(t)\xi, V(t)\xi) - 2||A^*(t)V(t)\xi||^2 - 2C_2^2||A^*(t)U(t)\xi||^2.$$

Because of $A^*(t)$ is invertible we have

$$2(D(t)U(t)\xi, V(t)\xi) \quad \leq$$
$$\leq 0.5||(A^*(t))^{-1}||^2||D(t)||^2||U(t)\xi||^2 + 2||(A^*(t))^{-1}||^{-2}||V(t)\xi||^2 \leq$$
$$\leq 0.5||(A^*(t))^{-1}||^4||D(t)||^2||A^*(t)U(t)\xi||^2 + 2||A^*(t)V(t)\xi||^2 \leq$$
$$\leq 2||A^*(t)V(t)\xi||^2 + 2C_2^2||\Lambda_0||^2||A^*(t)U(t)\xi||^2 \ .$$

Thus the inequality (29) is proved. □

Then we consider the problem

$$\lambda'(t) + \lambda(t) + \Lambda(t)\lambda(t) = 0, \quad \lambda(0) = \lambda_0 \ , \tag{33}$$

where $\lambda(t)$ is H-valued function on $[0, +_1)$, and $\Lambda(t)$ is a given continuous operator-valued function on $[0, +_1)$.

Lemma 6 *Let*

$$||\Lambda(t)|| \leq \epsilon_1, \quad for \quad t \in [0, T], \quad \epsilon_1 \in (0, 1) \ . \tag{34}$$

Then there exists a unique solution of the problem (33) on $[0, T]$ *and*

$$||\lambda(t)|| \leq \lambda_0 e^{-(1-\epsilon_1)t} \quad for \quad t \in [0, T] \ . \tag{35}$$

PROOF. (35) immediatly follows behind (33) and

$$\frac{\mathrm{d}}{\mathrm{dt}}(\lambda(t), \lambda(t)) = -2(\lambda(t), \lambda(t)) - 2\mathrm{Re}(\Lambda(t)\lambda(t), \lambda(t)) \leq$$
$$\leq -2(1 - \epsilon_1)(\lambda(t), \lambda(t)) \ .$$

□

Now we can prove the Theorem 3. From Condition B we get the existense of a local continuous solution of (16) near every t if $x(t) \in B(x_0, r||\varphi(x_0)||)$. Let us show that $x(t) \in B(x_0, r||\varphi(x_0)||)$ for all $[0, +\infty)$. In fact let $x(t_1)$ be the first intersection point with the boundary of the ball $B(x_0, r||\varphi(x_0)||)$ for the curve $x(t)$. So

$$||x(t) - x_0|| < r||\varphi(x_0)||, \quad \text{for} \quad t \in [0, t_1) \ ,$$

and

$$||x(t_1) - x_0|| = r||\varphi(x_0)|| \ .$$

Denote $A(t) = \rho\varphi'(x(t)), \quad C_1 = \max\{||Y_0||, ||\varphi'^{*-1}||_r\}$. From Lemma 4 we get the estimate

$$||Y(t)|| \quad \leq C_1 \quad \text{for} \quad t \in [0, t_1] \ ,$$

for the solution of the problem (16), where

$$C_1 = \max\{||Y_0||, ||\varphi'^{*-1}||_r\} \ .$$

The last term in (15) satisfies the estimate

$$\begin{aligned} ||(\varphi''(x(t))x'(t))Y(t)|| \quad &\leq \quad ||\varphi''(x(t))x'(t)||\,||Y(t)|| \quad \leq \\ \leq \quad ||\varphi''||_r||x'(t)||\,||Y(t)|| \quad &\leq \quad C_1^2||\varphi''||_r||\varphi||_r \ . \end{aligned} \tag{36}$$

It follows behind (18) that if $\rho > \rho_0$, then the conditions of Lemma 5 are valid for $C_2 = \max\{||\varphi'(x_0)Y_0 - E||, \epsilon\}$. At the same time, from (17) we obtain $C_2 < 1$. Further from Lemmas 5 and 6 and from (17) we get that for $\rho > \rho_0$ the solution of the problem (15) satisfies the inequalities

$$||\Lambda(t)|| \leq C_2 \quad \text{for} \quad t \in [0, t_1] \ ,$$

$$||\lambda(t)|| \leq \lambda_0 e^{-(1-C_2)t} \quad \text{for} \quad t \in [0, t_1] \ .$$

So we have the estimate for the solution of the problem (16)

$$||x(t_1) - x_0|| \leq ||\int_0^{t_1} x'(s)ds|| \leq C_1\,||\varphi(x_0)||\int_0^{t_1} e^{-(1-C_2)s}ds =$$

$$= \frac{C_1}{(1-C_2)}\,||\varphi(x_0)||(1 - e^{-(1-C_2)t_1}) < r||\varphi(x_0)|| \ .$$

The last one is in contradiction with the assumption that $x(t_1)$ belongs to the boundary of $B(x_0, r||\varphi(x_0)||)$. Thus $x(t) \in B(x_0, r||\varphi(x_0)||)$ for all $t \in [0, +\infty)$.

Consequently,

$$\lim_{t\to\infty} \varphi(x(t)) = \lim_{t\to\infty} \lambda(t) = 0$$

and $\lim_{t\to\infty} x(t)$ is the solution of the problem (1).

5 Numerical example

Let us consider the Cauchy problem for the following Kirchhoff Equation:

$$u_{tt}(t,x) - (a + b\int\limits_0^{2\pi} u_x^2(t,x)dx)u_{xx}(t,x) = 0, \quad t > 0, \quad x \in [0, 2\pi], \tag{37}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{38}$$

with the periodic initial data

$$u_0(x), u_1(x) \in C^\infty[0, 2\pi], u_0(0) = u_0(2\pi), u_1(0) = u_1(2\pi). \tag{39}$$

One can easily prove that the solution of the problem is periodic by x too. Denote

$$\begin{aligned} &\mu(t) = a + b\int\limits_0^{2\pi} u_x^2(t,x)dx, \quad u_1(t,x) = u_t(t,x)\ , \\ &u_2(t,x) = u_x(t,x), \quad u_2(x) = u_0'(x)\ , \end{aligned} \tag{40}$$

and write down the problem in form of the following system:

$$\begin{cases} \frac{\partial u_1}{\partial t} - \mu(t)\frac{\partial u_2}{\partial x} = 0, \quad u_1(0,x) = u_1(x), \\ \frac{\partial u_1}{\partial t} - \frac{\partial u_1}{\partial x} = 0, \quad u_2(0,x) = u_2(x)\ . \end{cases} \tag{41}$$

After the integration of this equations from t_0 to t, we get the system of integrodifferential equations

$$\begin{cases} u_1(t,x) - u_1(t_0,x) - \int\limits_{t_0}^{t} \mu(\tau)\frac{\partial u_2}{\partial x}(\tau,x)d\tau = 0 \\ u_2(t,x) - u_2(t_0,x) - \int\limits_{t_0}^{t} \frac{\partial u_2}{\partial x}(\tau,x)d\tau = 0\ . \end{cases} \tag{42}$$

Now we make the discretization of these equations by t, assuming $h = t - t_0$ to be small enough and the functions $u_1(t_0, x)$, $u_2(t_0, x)$ to be known . For $u_1(t, x)$ and $u_2(t, x)$ we get the following nonlinear system

$$\varphi(t, x) = \begin{cases} \varphi_1(t, x) = 0 \\ \varphi_2(t, x) = 0 \ , \end{cases} \tag{43}$$

where

$$\varphi_1(t, x) = u_1(t, x) - \frac{h}{2}\mu(t)\frac{\partial u_2}{\partial x}(t, x) - u_1(t_0, x) - \frac{h}{2}\mu(t_0)\frac{\partial u_2}{\partial x}(t_0, x),$$

$$\varphi_2(t, x) = u_2(t, x) - \frac{h}{2}\frac{\partial u_1}{\partial x}(t, x) - u_2(t_0, x) - \frac{h}{2}\frac{\partial u_1}{\partial x}(t_0, x).$$

Denote by φ a nonlinear operator acting from $C^1[0, 2\pi] \times C^1[0, 2\pi]$ to itself according to (43). The Fréchet derivative of the operator φ can be represented as a matrix operator

$$\varphi' = \begin{pmatrix} I & \varphi'_1 \\ \varphi'_2 & I \end{pmatrix} \ , \tag{44}$$

here φ'_1 and φ'_2 are linear operators: for $\xi \in C^1[0, 2\pi]$

$$\varphi'_1\xi(x) = -0.5h(\mu(t)\xi'(x) + \mu(u_2, \xi)\frac{\partial u_2}{\partial x}(t, x)) \ , \tag{45}$$

$$\varphi'_2\xi = -0.5h\xi' \ , \tag{46}$$

where

$$\mu(u, \xi) = 2b\int_0^{2\pi} u(x)\xi(x)dx \ . \tag{47}$$

Let us consider the Hilbert space $L^2_2[0, 2\pi]$ of pairs of real-valued functions (ξ_1, ξ_2), which are L^2-integrable on $[0, 2\pi]$, with the scalar product

$$< (\xi_1, \xi_2), (\eta_1, \eta_2) >= \int_0^{2\pi} (\xi_1(x)\eta_1(x) + \xi_2(x)\eta_2(x))dx \ . \tag{48}$$

The operator

$$\varphi'^* = \begin{pmatrix} I & \varphi'^*_1 \\ \varphi'^*_2 & I \end{pmatrix} \tag{49}$$

is adjoint to φ', where

$$\varphi'^{*}_{1}\xi(x) = 0.5h(\mu(t)\xi'(x) - \mu(\frac{\partial u_2}{\partial x}, \xi)u_2) \ , \tag{50}$$

$$\varphi'_2\xi(x) = 0.5h\xi'(x) \ . \tag{51}$$

For the numerical solving of the equation (16) using the scheme described in Sect.2 we calculate operators

$$\varphi'\varphi'^{*} = \begin{pmatrix} I + \varphi'_1\varphi'^{*}_2 & \varphi'_1 + \varphi'^{*}_1 \\ \varphi'_2 + \varphi'^{*}_2 & I + \varphi'_2\varphi'^{*}_1 \end{pmatrix} \ , \tag{52}$$

$$\varphi'^{*}\varphi' = \begin{pmatrix} I + \varphi'^{*}_1\varphi'_2 & \varphi'_1 + \varphi'^{*}_1 \\ \varphi'_2 + \varphi'^{*}_2 & I + \varphi'^{*}_2\varphi'_1 \end{pmatrix} \ , \tag{53}$$

where

$$(\varphi'_1 + \varphi'^{*}_1)\xi(x) = 0.5h[(1-\mu(t))\xi'(x) - \mu(u_2, \xi)u_{2x}] \ ,$$

$$(\varphi'_2 + \varphi'^{*}_2)\xi(x) = -0.5h[(1-\mu(t))\xi'(x) + \mu(u_2, \xi)u_{2x}] \ ,$$

$$\begin{aligned} \varphi'_1\varphi'^{*}_2\xi(x) &= -0.25h^2[\mu(t)^2\xi''(x) - \mu(t)\mu(u_{2x}, \xi)u_{2x} + \\ &+\mu(t)\mu(u_2, \xi')u_{2x} - \mu(u_{2x}, \xi)\mu(u_2, u_2)u_{2x}] = \\ &= -0.25h^2[\mu(t)^2\xi''(x) - \mu(t)\mu(u_{2x}, \xi)u_{2x} - \mu(u_{2x}, \xi)\mu(u_2, u_2)u_{2x}], \end{aligned}$$

$$\varphi'_1\varphi'^{*}_2\xi(x) = -0.25h^2[\mu(t)^2\xi''(x) + 2(a - 2\mu(t))\mu(u_{2x}, \xi)u_{2x}],$$

$$\varphi'_2\varphi'^{*}_1\xi(x) = \varphi'^{*}_1\varphi'_2\xi(x) = -0.25h^2\xi''(x),$$

$$\begin{aligned} \varphi'^{*}_2\varphi'_1\xi(x) = -0.25h^2[\mu(t)^2\xi''(x) + \mu(u_2, \xi)u_{2xx} + \\ +\mu(t)\mu(u_{2xx}, \xi)u_2 - \mu(u_{2x}, u_{2x})\mu(u_2, \xi)u_2]. \end{aligned}$$

For the calculation of $\varphi'^{*}_2\varphi'_1\xi$ one can use the relation

$$\mu(u_{2xx}, \xi) = -\mu(u_{2x}, \xi') \ .$$

One can obtain the initial approximation Y_0 from the following operator equations

$$\begin{pmatrix} I & \varphi'_1 \\ \varphi'_2 & I \end{pmatrix} \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \ , \tag{54}$$

so

$$Y_{11} = (I - \varphi_1'\varphi_2')^{-1}, \quad Y_{21} = -\varphi_2' Y_{11},$$

$$Y_{22} = (I - \varphi_2'\varphi_1')^{-1}, \quad Y_{12} = -\varphi_1' Y_{22},$$

where

$$\varphi_1'\varphi_2'\xi(x) = 0.25h^2[\mu(t)\xi''(x) - \mu(u_{2x},\xi)u_{2x}] \ ,$$

$$\varphi_2'\varphi_1'\xi(x) = 0.25h^2[\mu(t)\xi''(x) + \mu(u_2,\xi)u_{2xx}] \ .$$

For the testing of the algorithm the problem (37-38) was numerically solved for $(x,t) \in [0,2\pi] \times [0,2\pi]$, $a = 1$, $b = 2$, $u_1(x) \equiv 0$ and

$$u_0(x) = 0 \quad \text{for} \quad x \in [0,\frac{\pi}{2}] \bigcup [\frac{3\pi}{2}, 2\pi] \ ,$$

$$u_0(x) = 10\exp[\frac{1}{\frac{\pi}{2} - x} + \frac{1}{x - \frac{3\pi}{2}}] \quad \text{for} \quad x \in (\frac{\pi}{2}, \frac{3\pi}{2}) \ .$$

To this end the problem (16) was solved for every $t = 0.1hl, \quad l = 1,\ldots,N$. As an initial iteration step, $\tau_0 = 0.1$ was choised, and then for every next iteration the value of the step was determined by the recursion formula ([8]) $\tau_k = \delta_{k-1}\tau_{k-1}/\delta_k$, where the discrepancy δ_k after the iteration number k was calculated by the following formula

$$\delta_k = \sup_{x\in[0,2\pi]} \sqrt{\varphi_1^2(t,x) + \varphi_2^2(t,x)} \ .$$

For every t the iterations were done until the discrepancy became less than 10^{-3}. Usually it became after 8–10 iterations.

The results of the numerical calculations for $t \in [0,\pi/2]$ are illustrated by the following picture.

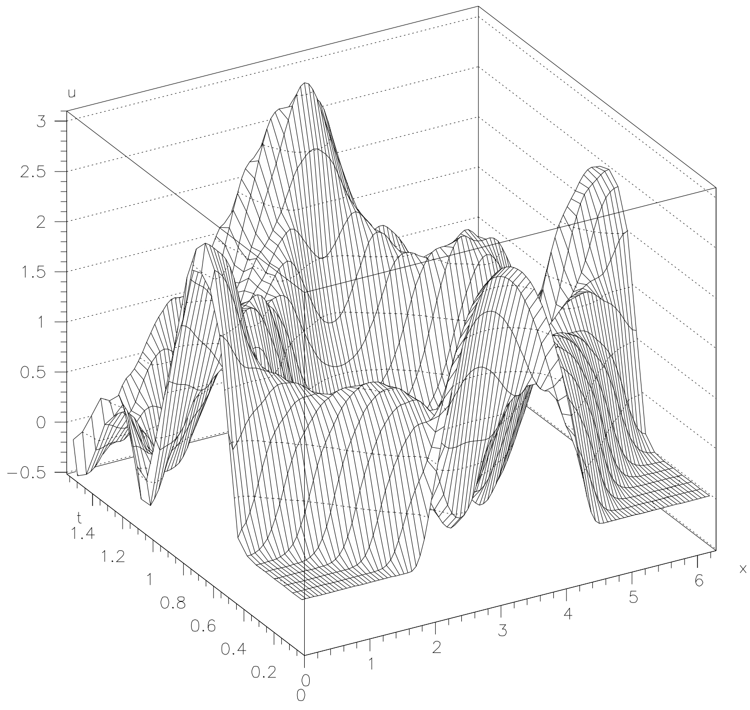

Fig. 1. The solution of the problem for $x \in [0, 2\pi]$ and $t \in [0, \pi/2]$

6 Conclusion

In the present paper a modification of the Continuous Analogy of Newton Method for the numerical solving of the nonlinear problems is suggested. It permits one to replace the inversion of the derivative operator $\varphi'(x)$ on every step of iterations to its inversion only in the initial approximation point x_0. Iterative process "keeps" determined initial $Y_0 = {\varphi'}^{-1}(x_0)$, since $Y(x)$ remains nearby ${\varphi'}^{-1}(x)$ during the all iterations. At the same time the convergence theorem is proved under almost the same conditions as for CANM.

Numerical calculations for the model problems have shown the effectiveness and adequately fast convergence of the iterative schemes based on the suggested method. The example of the Kirchhoff equation has shown how one can considerably diminish the quantity of the calculations, avoiding the laborious procedure of the multiplication of large size matrices by means of some preliminary analytical calculations. On the other hand, we would like to mention that the method makes it possible to construct the parallelzing algorithms.

It is clear from the Theorem 3 that the convergence of iterations depends on the choice of ρ in (16). However ρ_0 is a priori unknown in numerical realizations. So we started from $\rho = 1$ and, in the case of divergence of the process, repeated it with a larger value of ρ and probably with a larger number of iterations. However for considered nunerical examples, the convergence had usualy already occured for $\rho = 1$.

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