Integral Transforms and Special Functions

ISOMETRIC PROPERTIES OF THE HANKEL TRANSFORMATION IN WEIGHTED SOBOLEV SPACES

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ABSTRACT. It is shown that the Hankel transformation H_ν acts in a class of weighted Sobolev spaces. Especially, the isometric mapping property of H_ν which holds on $L^2(\mathbb{R}_+, rdr)$ is extended to spaces of arbitrary Sobolev order. The novelty in the approach consists in using techniques developed by B.-W. Schulze and others to treat the half-line $\mathbb{R}_+$ as a manifold with a conical singularity at $r = 0$. This is achieved by pointing out a connection between the Hankel transformation and the Mellin transformation. The procedure proposed leads at the same time to a short proof of the Hankel inversion formula. An application to the existence and higher regularity of solutions, including their asymptotics, to the 1+1 dimensional edge-degenerate wave equation is given.

1. INTRODUCTION

For $\nu \geq 0$, we consider the Hankel transformation of order ν,

$$H_\nu u(\xi) = \int_0^\infty J_\nu(\xi r) u(r) r\, dr \tag{1}$$

where $J_\nu(r)$ is the Bessel function of the first kind. A classical result says that H_ν induces an isometry

$$H_\nu \colon L^2(\mathbb{R}_+, rdr) \to L^2(\mathbb{R}_+, rdr).$$

Moreover, H_ν is self-inverse on $L^2(\mathbb{R}_+, rdr)$, by the Hankel inversion theorem.

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The Hankel transformation in different function spaces has been studied by several authors (see, e.g., [4, 8, 14, 15, 17]). The major goal of this contribution is to extend the Hankel transformation H_ν by continuity to a class of weighted Sobolev spaces $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$, defined in (32), where $\gamma > -3/2 - \nu$, and to study the properties of H_ν on these spaces. Especially, $L^2(\mathbb{R}_+, rdr) = \mathcal{H}^{0,-1/2}(\mathbb{R}_+)$, and H_ν induces a self-inverse isometry from $\mathcal{H}^{s,-1/2}(\mathbb{R}_+)$ onto itself, for any $s \in \mathbb{R}$.

The results mentioned are obtained by utilizing a connection between the Hankel transformation and the Mellin transformation, M. The starting point is the observation that, for $u \in C_0^\infty(\mathbb{R}_+)$, we have

$$|MH_\nu u(z)| \leq C_\delta \langle z \rangle^\delta |MIu(z)| = C_\delta \langle z \rangle^\delta |Mu(2-z)| \quad \text{if } \operatorname{Re} z = 1 - \delta \tag{2}$$

for a certain constant $C_\delta > 0$, and any $\delta \in \mathbb{R}$, $\delta > -1 - \nu$. Here $\langle z \rangle = (1 + |z|^2)^{1/2}$ and the inversion I is defined by

$$Iu(r) = r^{-2} u(r^{-1}). \tag{3}$$

In particular, $C_\delta = 1$ for $\delta = 0$ such that

$$|MH_\nu u(z)| = |Mu(\bar{z})| \quad \text{if } \operatorname{Re} z = 1,$$

by replacing u by $H_\nu u$ in (2). Investigating these phenomena more closely, it turns out that the Hankel transformation H_ν as operator acting from $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ to $\mathcal{H}^{s-\gamma-1/2,-1-\gamma}(\mathbb{R}_+)$ with $\gamma > -3/2 - \nu$ can be rewritten as

$$H_\nu = I \operatorname{op}_M^{1/2-\gamma}(a_\nu) = \operatorname{op}_M^{3/2+\gamma}(b_\nu) I \tag{4}$$

i.e., as composition of the Mellin operator $\operatorname{op}_M^{1/2-\gamma}(a_\nu)$ with symbol

$$a_\nu(z) = 2^{1-z} \frac{\Gamma((\nu - z)/2 + 1)}{\Gamma((\nu + z)/2)} \tag{5}$$

and the inversion I, and likewise as composition of the inversion I and the Mellin operator $\operatorname{op}_M^{3/2+\gamma}(b_\nu)$ with symbol

$$b_\nu(z) = 2^{z-1} \frac{\Gamma((\nu + z)/2)}{\Gamma((\nu - z)/2 + 1)}. \tag{6}$$

Notice the fundamental relation

$$a_\nu(z)^{-1} = b_\nu(z) = a_\nu(2-z) \tag{7}$$

which in its consequence, e.g., yields the Hankel inversion theorem. Notice further that $a_\nu(z)$ is meromorphic on $\mathbb{C}$ with simple poles at $z = \nu + 2j + 2$, $j \in \mathbb{N}$, while $b_\nu(z)$ is meromorphic on $\mathbb{C}$ with simple poles at $z = -\nu - 2j$, $j \in \mathbb{N}$. The first pole for $a_\nu(z)$ appearing at $z = \nu + 2$ (or the first pole for $b_\nu(z)$ appearing at $z = -\nu$) is responsible for the impossibility to explain the Hankel transformation H_ν on $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$, $\gamma \le -3/2 - \nu$.

It is seen that the situation under consideration is quite symmetric with respect to the conormal order $\gamma = -1/2$. This corresponds to the weight line $\Gamma_1 = \{z \in \mathbb{C}; \operatorname{Re} z = 1\}$ in the Mellin image. For that reason, we shall henceforth write $-1/2 + \delta$ instead of γ. Then $1/2 - \gamma$ and $3/2 + \gamma$ in (4) become resp. $1 - \delta$ and $1 + \delta$.

Let us discuss an illuminating example.

Example 1. Let $p \in \mathbb{C}$, $|\operatorname{Re} p| < 1 + \nu$. Then, for $\delta, \delta' \in \mathbb{R}$, $\delta < \delta'$, we have $r^{p-1} \in \mathcal{H}^{\infty,-1/2+\delta}(\mathbb{R}_+) + \mathcal{H}^{\infty,-1/2+\delta'}(\mathbb{R}_+)$ if and only if $\delta < \operatorname{Re} p < \delta'$. Now choosing δ, δ' in such a way that $\delta > -1 - \nu$, $-\delta' > -1 - \nu$, we see that the Hankel transformation of order ν is applicable and, using a table, e.g., [3, 6.8(1), 8.5(7)], [7, 2.12.2.2], we find that

$$H_\nu\left\{r^{p-1}\right\}(\xi) = b_\nu(p+1)\,\xi^{-(p+1)}. \tag{8}$$

This can be refined as follows. Choose a cut-off function $\omega \in C^\infty(\overline{\mathbb{R}}_+)$, i.e., $\omega = 1$ near $r = 0$ and $\omega = 0$ for large r, and suppose that $\operatorname{Re} p > -1 - \nu$ instead of $|\operatorname{Re} p| < 1 + \nu$. Then

$$H_\nu\left\{\omega(r) r^{p-1}\right\} \in \begin{cases} \mathcal{Z}_\nu(\mathbb{R}_+) \oplus I\mathcal{E}_P(\mathbb{R}_+) & \text{if } p \notin 1 + \nu + 2\mathbb{N}, \\ \mathcal{Z}_\nu(\mathbb{R}_+) & \text{if } p \in 1 + \nu + 2\mathbb{N}. \end{cases}$$

Here $\mathcal{Z}_\nu(\mathbb{R}_+)$ is Zemanian's space which provides the asymptotic behaviour near $r = 0$, cf. Lemma 5, and $I\mathcal{E}_P(\mathbb{R}_+)$ with $P = \{(-p+1, 0)\}$ gives the asymptotics at $r = \infty$; $\mathcal{E}_P(\mathbb{R}_+)$ abbreviates $\operatorname{span}\{\omega(r) r^{p-1}\}$. Moreover, $I\mathcal{E}_P(\mathbb{R}_+) \subset \bigcap_{\delta > -\operatorname{Re} p} \mathcal{H}^{\infty,-1/2+\delta}(\mathbb{R}_+)$ such that the Hankel transformation is applicable to $I\mathcal{E}_P(\mathbb{R}_+)$ if we choose $\delta > -1 - \nu$. Compared with the Hankel inversion theorem, however, H_ν is displaced by certain $H_\nu^{(j)}$ provided that $-1-\nu-2j<\delta<1-\nu-2j$ for $j \in \mathbb{N}$, $j \geq 1$, as we will see soon.

A basic property of the Hankel transformation H_ν is that it diagonalizes the differential operator

$$L_\nu = -r^{-2}\left(r\frac{\partial}{\partial r}\right)^2 + r^{-2}\nu^2, \tag{9}$$

i.e., we have

$$H_\nu L_\nu = \xi^2 H_\nu, \tag{10}$$

where ξ^2 means multiplication in the image space of H_ν. Making use of (4), this property is easily verified, e.g., on the space $\mathcal{H}^{-\infty,1/2}(\mathbb{R}_+)$:

$$\begin{aligned} H_\nu L_\nu &= I\operatorname{op}_M^2(a_\nu) r^{-2}\operatorname{op}_M^0(l_\nu) = \\ &= I\xi^{-2}\operatorname{op}_M^0(T^2 a_\nu)\operatorname{op}_M^0(l_\nu) = \xi^2 I\operatorname{op}_M^0((T^2a_\nu)l_\nu) = \xi^2 I\operatorname{op}_M^0(a_\nu) = \xi^2 H_\nu, \end{aligned}$$

because of $(T^2a_\nu)(z)\, l_\nu(z) = a_\nu(z)$ for

$$2^{-1-z}\frac{\Gamma((\nu-z)/2)}{\Gamma((\nu+z)/2+1)}(-z^2+\nu^2) = 2^{1-z}\frac{\Gamma((\nu-z)/2)(\nu-z)/2}{\Gamma((\nu+z)/2+1)/(\nu+z/2)},$$

where $(T^2a_\nu)(z) = a_\nu(z+2)$ and $l_\nu(z) = -z^2+\nu^2$ is the Mellin symbol of r^2L_ν.

We shall employ property (10) to discuss the solvability of the edge-degenerate wave equation in 1+1 dimensions,

$$\partial_t^2 u + L_\nu u = f(t) \text{ on } \mathbb{R}\times\mathbb{R}_+, \tag{11}$$

where the given right-hand side f fulfils $f=0$ for $t<0$. The wave operator $\partial_t^2 + L_\nu$ mainly becomes a constant coefficient operator having the symbol $-\tau^2+\xi^2$ when viewed with respect to the Fourier transformation in t and the Hankel transformation in r. Thus it is possible to investigate (11) without decomposing its solution into differently behaved pieces, near resp. $r=0$ and $r=\infty$. Moreover, the operator $\partial_t^2+L_\nu$ serves as model for an operator on a configuration having conical singularities in the spatial variables that is strictly hyperbolic away from the set of singularities. In our example, the conical singularity is at $r=0$ on $\mathbb{R}_+$; then $\{(t,r)\in\mathbb{R}\times\mathbb{R}_+; r=$

$0\}$ becomes the set of singularities. This set is identified as an *edge* by the *typical degeneracy* of the differential operator $\partial_t^2 + L_\nu$ near it. This is referred to as to be edge-degenerate, cf. [9, Chapter 3]. We shall prove that Eq. (11) possesses a unique solution u satisfying $u = 0$ for $t < 0$ in the wedge Sobolev space $\mathcal{W}_\eta^{s+1,1/2}(\mathbb{R} \times \mathbb{R}_+)$ provided that $f \in \mathcal{W}_\eta^{s,-3/2}(\mathbb{R} \times \mathbb{R}_+)$. Further we shall discuss the asymptotics of these solutions near $r = 0$.

The main references for us concerning the Hankel transformation are E. C. Titchmarsh [14] and A. H. Zemanian [17]. Let us remark that while for these authors the Hankel transformation was given by the integral in (1) we look at it mainly as being the operator (4). The techniques utilized to treat the point $r = 0$ as a conical singularity for the half-line $\mathbb{R}_+$ are borrowed from B.-W. Schulze [9], [11].

The paper is organized as follows. In Section 2 we state our main results concerning the properties of the Hankel transformation. These results are proved in Section 3. In Section 4 we further recapitulate some results of A. H. Zemanian looked at from our point of view on the Hankel transformation. Finally, in Section 5, we apply the results previously obtained to study the 1+1 dimensional edge-degenerate wave equation. In an appendix, the reader finds various definitions of functional spaces used throughout, including spaces with asymptotics, and the convention of Mellin operators.

2. Statement of the Results

In this section we state our main results concerning the properties of the Hankel transformation. These results are then proved in Section 3. The Hankel transformation H_ν is extended to a class of weighted Sobolev spaces $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$, the definition of which can be found in the appendix, see (32). The various extensions that coincide on intersections are again denoted by H_ν. Especially, it makes sense to give a meaning to H_ν on sums of such spaces, which becomes important when defining the Hankel transformation on $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$, see Lemma 7.

We introduce the following partition of the real line $\mathbb{R}$: for $j \in \mathbb{Z}$, we put

$$I_j = \begin{cases} (-1+2j+\nu, 1+2j+\nu) & \text{if } j \geq 1, \\ (-1-\nu, 1+\nu) & \text{if } j = 0, \\ (-1+2j-\nu, 1+2j-\nu) & \text{if } j \leq -1. \end{cases}$$

If we wish to express the dependence on ν explicitely, then we write $I_j = I_j(\nu)$. We have $\mathbb{R} = \bigcup_{j\in\mathbb{Z}} \overline{I_j}$, whereas, for j, $l \in \mathbb{Z}$, $j \neq l$, $\overline{I_j} \cap \overline{I_l}$ consists of one point, if $|j-l| = 1$, and is empty otherwise.

Theorem 1 (Main Theorem). *Let $s \in \mathbb{R}$, $\delta \in I_0 \cup \bigcup_{j\geq 1} \overline{I_j}$, i.e., $\delta > -1-\nu$.*

(a) *The Hankel transformation H_ν, defined on $C_0^\infty(\mathbb{R}_+)$ by* (1)*, extends by continuity to*

$$H_\nu \colon \mathcal{H}^{s,-1/2+\delta}(\mathbb{R}_+) \to \mathcal{H}^{s-\delta,-1/2-\delta}(\mathbb{R}_+) \tag{12}$$

(b) *For $\delta \in I_0 \cup \bigcup_{j\geq 1} I_j$, i.e., $\delta \notin 1+\nu+2\mathbb{N}$, H_ν given by* (12) *is an isomorphism.*

(c) *For $\delta \in \overline{I_j} \cap \overline{I_{j+1}}$ for some $j \in \mathbb{N}$, i.e., $\delta \in 1+\nu+2\mathbb{N}$, H_ν given by* (12) *is injective, but not surjective, with dense range.*

(d) *For $\delta \in I_0$, i.e., $|\delta| < 1+\nu$, the inverse to H_ν is again H_ν.*

(e) *For $\delta = 0$, H_ν is even an isometry.*

Remark 1. The statements about continuity in (12) extend to $H_\nu \colon \mathcal{H}^{\pm\infty,-1/2+\delta}(\mathbb{R}_+) \to \mathcal{H}^{\pm\infty,-1/2-\delta}(\mathbb{R}_+)$. This also concerns the statements in (b), (c), and (d).

For $\delta \in I_j$, where $j \in \mathbb{N}$, we denote the inverse to H_ν as operator defined by (12) by $H_\nu^{(j)}$. Then $H_\nu^{(j)}$ is actually independent of $\delta \in I_j$ in the sense explained above for H_ν. For $j = 0$, we have $H_\nu^{(0)} = H_\nu$, by Theorem 1(d).

Theorem 2. *Let j, $l \in \mathbb{N}$, $j < l$. Then, for $\delta \in I_j$, $\delta' \in I_l$, and any $u \in \mathcal{H}^{-\infty,-1/2-\delta}(\mathbb{R}_+) \cap \mathcal{H}^{-\infty,-1/2-\delta'}(\mathbb{R}_+)$, we have that*

$$H_\nu^{(j)} u - H_\nu^{(l)} u = \sum_{k=j}^{l-1} c_k(u)\, \xi^{\nu+2k} \tag{13}$$

where the functionals c_k are given by

$$c_k(u) = \frac{(-1)^k}{k!} \frac{2^{-\nu-2k}}{\Gamma(\nu+k+1)} \tilde{u}(\nu+2k+2). \tag{14}$$

These functionals are continuous on $\big(\mathcal{H}^{-\infty,-1/2-\delta}(\mathbb{R}_+) \cap \mathcal{H}^{-\infty,-1/2-\delta'}(\mathbb{R}_+)\big)'$.

Introducing the "truncated" Bessel function

$$J_\nu^{(l)}(r) = \left(\frac{r}{2}\right)^\nu \sum_{k=l}^{\infty} \frac{(-1)^k}{k!} \frac{1}{\Gamma(\nu+k+1)} \left(\frac{r}{2}\right)^{2k}, \quad r \in \mathbb{R}_+,$$

i.e., $J_\nu^{(l)}(r)$ is obtained via subtracting off the first l terms in the series expansion of $J_\nu(r)$, and using (13), (14) with $j = 0$, a brief calculation shows that, for $l \in \mathbb{N}$, we have, supplementary to (1), the following truncated Hankel transformation (see also [4, 8, 15]):

$$H_\nu^{(l)} u(\xi) = \int_0^\infty J_\nu^{(l)}(\xi r) u(r) r\, dr, \quad u \in C_0^\infty(\mathbb{R}_+). \tag{15}$$

Closing we determine the inverse to H_ν that corresponds to the situation described in Theorem 1(c). The densely defined operator thus obtained is denoted by $H_\nu^{(j\to j+1)}$.

Theorem 3. *Let $s \in \mathbb{R}$, $j \in \mathbb{N}$, and $\delta = 1 + \nu + 2j$. Then the range of H_ν as operator from $\mathcal{H}^{s,-1/2+\delta}(\mathbb{R}_+)$ to $\mathcal{H}^{s-\delta,-1/2-\delta}(\mathbb{R}_+)$ consists of all functions $u \in \mathcal{H}^{s-\delta,-1/2-\delta}(\mathbb{R}_+)$ for which $\tilde{u}(z)/(z-(\nu+2j+2))$ is square-integrable on $\{z \in \mathbb{C}; \operatorname{Re} z = 1+\delta\}$ close to $z = \nu+2j+2$.*

If $u \in C_0^\infty(\mathbb{R}_+)$, then u belongs to this range if and only if $\tilde{u}(\nu+2j+2) = 0$. In this case the action of $H_\nu^{(j\to j+1)}$ on u is given by (15) with l replaced by either j or $j+1$.

3. Proofs

We prove Theorems 1–3 in a series of lemmas. The basic estimate is provided in Lemma 1. Then, in Lemma 2 and Lemma 3, it is shown that the operators $I \operatorname{op}_M^{1-\delta}(a_\nu) = \operatorname{op}_M^{1+\delta}(b_\nu) I$ replacing H_ν are resp. inverse to each other and isometries when $\delta = 0$. Finally, in Lemma 4, the operators H_ν and $I \operatorname{op}_M^1(a_\nu) = \operatorname{op}_M^1(b_\nu) I$ are identified. This immediately extends to values of $\delta > -1-\nu$ due to holomorphy of a_ν in the corresponding region. All together, Theorem 1 is implied. Theorem 2 and Theorem 3 equally follow by a simple calculation. For the functional spaces used and notation of Mellin operators, see the appendix.

Lemma 1 (The basic estimate)**.** *Let $\chi \in C^\infty(\mathbb{C})$ be an excision function for the poles of $a_\nu(z)$. Then, for any $\delta \in \mathbb{R}$, we have the estimate*

$$\sup_{\operatorname{Re} z = 1-\delta} (1+|z|^2)^{-\delta/2} |\chi(z)\, a_\nu(z)| < \infty. \tag{16}$$

Similarly, for $b_\nu(z)$ we have the estimate

$$\sup_{\operatorname{Re} z = 1+\delta} (1+|z|^2)^{-\delta/2} |\chi(2-z)\, b_\nu(z)| < \infty. \tag{17}$$

Moreover, these estimates hold uniformly for $\delta \in [\delta_0, \delta_1]$, δ_0, $\delta_1 \in \mathbb{R}$, $\delta_0 < \delta_1$. In addition,

$$|a_\nu(z)| = |b_\nu(z)| = 1 \text{ for } \operatorname{Re} z = 1. \tag{18}$$

Thereby, being an excision function for the poles of $a_\nu(z)$ means

$$\chi(z) = \begin{cases} 1 & \text{if } |z-\nu-2j-2| \geq 2\epsilon \text{ for all } j \in \mathbb{N} \\ 0 & \text{if } |z-\nu-2j-2| \leq \epsilon \quad \text{for some } j \in \mathbb{N} \end{cases}$$

for a certain $\epsilon > 0$. Notice that then $\chi(2-z)$ is an excision function for the poles of $b_\nu(z)$.

The estimates in Lemma 1 basically mean that outside the poles of $a_\nu(z)$

$$|a_\nu(z)| \asymp (1+|z|^2)^{\delta/2} \text{ for } \operatorname{Re} z = 1-\delta,$$

and similarly for $b_\nu(z)$. Repeat the notation $\langle z \rangle = (1+|z|^2)^{1/2}$.

Proof of Lemma 1. We prove the estimate for $a_\nu(z)$, $b_\nu(z)$ simultaneously. Fix δ_0, $\delta_1 \in \mathbb{R}$, $\delta_0 < \delta_1$. Then

$$C_0 \langle z \rangle^\delta \leq \left| \frac{\Gamma((\nu+z)/2+\delta)}{\Gamma((\nu+z)/2)} \right| \leq C_1 \langle z \rangle^\delta \tag{19}$$

for $z \in \mathbb{C}$, $\operatorname{Re} z = 1-\delta$, $|\operatorname{Im} z| \geq \epsilon$, and $\delta \in [\delta_0, \delta_1]$, with some constants C_0, $C_1 > 0$ only depending on δ_0, δ_1, and $\epsilon > 0$. Indeed, $\Gamma((\nu+z)/2+\delta)/\Gamma((\nu+z)/2)$ behaves like a classical pseudo-differential symbol of order δ on $\operatorname{Re} z = 1-\delta$, since there we have an asymptotic expansion

$$\frac{\Gamma((\nu+z)/2+\delta)}{\Gamma((\nu+z)/2)} \sim z^\delta \sum_{s=0}^\infty \alpha_s z^{-s} \text{ as } \operatorname{Im} z \to \pm\infty$$

with coefficients α_s depending on δ, cf. [6, Chapter IV, §5.1].

Now let $z \in \mathbb{C}$, $\operatorname{Re} z = 1-\delta$. Then

$$\frac{\nu-\bar{z}}{2}+1=\frac{\nu+z}{2}+\delta$$

such that

$$\left|\frac{\Gamma((\nu-z)/2+1)}{\Gamma((\nu+z)/2)}\right|=\left|\frac{\Gamma((\nu-\bar{z})/2+1)}{\Gamma((\nu+z)/2)}\right|=\left|\frac{\Gamma((\nu+z)/2+\delta)}{\Gamma((\nu+z)/2)}\right|$$

for $\Gamma(z)=\overline{\Gamma(\bar{z})}$ and, by (19),

$$C_0'\langle z\rangle^{\delta}\leq\left|a_\nu(z)\right|=\left|b_\nu(z)^{-1}\right|\leq C_1'\langle z\rangle^{\delta}$$

for $\operatorname{Re} z = 1-\delta$, $|\operatorname{Im} z|\geq\epsilon$, with certain constants C_0', $C_1'>0$. □

The involution I defined in (3) satisfies $M(Iu)(z) = Mu(2-z)$ for $\operatorname{Re} z = 1-\delta$, $u \in \mathcal{H}^{-\infty,-1/2+\delta}(\mathbb{R}_+)$. Hence, for any s, $\delta\in\mathbb{R}$, it realizes an isometry

$$I\colon \mathcal{H}^{s,-1/2+\delta}(\mathbb{R}_+)\to\mathcal{H}^{s,-1/2-\delta}(\mathbb{R}_+).$$

Moreover, we have the relation $I\operatorname{op}_M^{1-\delta}(h)=\operatorname{op}_M^{1+\delta}(h^I)I$ on $\mathcal{H}^{-\infty,-1/2+\delta}(\mathbb{R}_+)$, where h is any Mellin symbol given on $\Gamma_{1-\delta}$, cf. (33), and $h^I(z)=h(2-z)$. The estimates (16), (17) allow us to define the Mellin operators

$$\begin{aligned}\operatorname{op}_M^{1-\delta}(a_\nu)&\colon \mathcal{H}^{s,-1/2+\delta}(\mathbb{R}_+)\to\mathcal{H}^{s-\delta,-1/2+\delta}(\mathbb{R}_+),\\ \operatorname{op}_M^{1+\delta}(b_\nu)&\colon \mathcal{H}^{s,-1/2-\delta}(\mathbb{R}_+)\to\mathcal{H}^{s-\delta,-1/2-\delta}(\mathbb{R}_+)\end{aligned}$$

for s, $\delta\in\mathbb{R}$, $\delta\notin -1-\nu-2\mathbb{N}$. Thus we can consider the compositions

$$I\operatorname{op}_M^{1-\delta}(a_\nu)=\operatorname{op}_M^{1+\delta}(b_\nu)I\colon \mathcal{H}^{s,-1/2+\delta}(\mathbb{R}_+)\to\mathcal{H}^{s-\delta,-1/2-\delta}(\mathbb{R}_+) \tag{20}$$

for $\delta\notin -1-\nu-2\mathbb{N}$.

Lemma 2. *The operators $I\operatorname{op}_M^{1-\delta}(a_\nu)$, $I\operatorname{op}_M^{1+\delta}(a_\nu)$ are inverse to each other provided that $\pm\delta\notin 1+\nu+2\mathbb{N}$.*

Proof. For $\pm\delta \notin 1+\nu+2\mathbb{N}$, we simply have

$$I \operatorname{op}_M^{1-\delta}(a_\nu) \operatorname{op}_M^{1-\delta}(b_\nu) I = I \operatorname{op}_M^{1-\delta}(a_\nu b_\nu) I = \operatorname{id}_{\mathcal{H}^{s-\delta,-1/2-\delta}(\mathbb{R}_+)},$$
$$\operatorname{op}_M^{1-\delta}(b_\nu) I^2 \operatorname{op}_M^{1-\delta}(a_\nu) = \operatorname{op}_M^{1-\delta}(b_\nu) \operatorname{op}_M^{1-\delta}(a_\nu) = \operatorname{id}_{\mathcal{H}^{s,-1/2+\delta}(\mathbb{R}_+)}.$$

□

Lemma 3. *For every $s \in \mathbb{R}$, the operator $I \operatorname{op}_M^1(a_\nu)$ is an isometry on $\mathcal{H}^{s,-1/2}(\mathbb{R}_+)$.*

Proof. From Lemma 2 we know that $I \operatorname{op}_M^1(a_\nu)$ is an isomorphism from $\mathcal{H}^{s,-1/2}(\mathbb{R}_+)$ onto itself. That $I \operatorname{op}_M^1(a_\nu)$ maps isometrically on this space follows from the facts that I is an isometry on $\mathcal{H}^{s,-1/2}(\mathbb{R}_+)$ and $\operatorname{op}_M^1(a_\nu)$ maps $\mathcal{H}^{s,-1/2}(\mathbb{R}_+)$ isometrically into itself by the relation (18). □

Lemma 4. *On the space $C_0^\infty(\mathbb{R}_+)$, we have*

$$H_\nu = I \operatorname{op}_M^1(a_\nu) = \operatorname{op}_M^1(b_\nu) I. \tag{21}$$

Proof. Let $u \in C_0^\infty(\mathbb{R}_+)$. Then, for $\operatorname{Re} z = 1$, we have

$$\begin{aligned} M(H_\nu u)(z) &= M_{\xi\to z}\Big(\int_0^\infty J_\nu(\xi r) u(r) r\, dr\Big) \\ &= \int_0^\infty r^{-z} b_\nu(z) u(r) r\, dr, \end{aligned}$$

since the improper integral $\int_0^\infty \xi^{z-1} J_\nu(\xi r)\, d\xi = r^{-z} b_\nu(z)$ converges uniformly in $r \in [a_0, a_1]$ for $0 < a_0 < a_1 < \infty$, cf. [3, 8.5(7)], such that the change of order of integration is justified. Thus we obtain that

$$M(H_\nu u)(z) = b_\nu(z) Mu(2-z) = b_\nu(z) M(Iu)(z) \text{ if } \operatorname{Re} z = 1,$$

i.e., actually $H_\nu u = \operatorname{op}_M^1(b_\nu) I u = I \operatorname{op}_M^1(a_\nu) u$. □

As a simple consequence of Lemma 4 we get that (21) extends to hold on $\mathcal{H}^{-\infty,-1/2}(\mathbb{R}_+)$, and then, for $\delta > -1-\nu$, even on $\mathcal{H}^{-\infty,-1/2+\delta}(\mathbb{R}_+)$ by taking into account that $a_\nu(z)$ is holomorphic in the region $\operatorname{Re} z < \nu + 2$. We have now collected enough material to conclude the proof of Theorem 1 besides (c) that should be finished together with the proof of Theorem 3.

Proof of Theorem 1(a), (b), (d), (e). We have mentioned that, for $\delta > -1-\nu$, $H_\nu = I\,\mathrm{op}_M^{1-\delta}(a_\nu) = \mathrm{op}_M^{1+\delta}(b_\nu)I$ holds as mappings from $\mathcal{H}^{-\infty,-1/2+\delta}(\mathbb{R}_+)$ to $\mathcal{H}^{-\infty,-1/2-\delta}(\mathbb{R}_+)$. The mapping property (12) is then concluded by the estimates in Lemma 1. This gives us (a). The assertions (b), (d) are implied by Lemma 2, and (e) follows from Lemma 3. $\square$

Proof of Theorem 2. Let $u \in C_0^\infty(\mathbb{R}_+)$. Then

$$\begin{aligned}\mathrm{op}_M^{1+\delta}(a_\nu)u - \mathrm{op}_M^{1+\delta'}(a_\nu)u &= \frac{1}{2\pi i}\int_{\Gamma_{1+\delta}} r^{-z} a_\nu(z)\tilde{u}(z)\,dz - \frac{1}{2\pi i}\int_{\Gamma_{1+\delta'}} r^{-z} a_\nu(z)\tilde{u}(z)\,dz \\ &= -\sum_{k=j}^{l-1} \mathrm{Res}_{z=\nu+2k+2}\; r^{-z} a_\nu(z)\tilde{u}(z) \\ &= \sum_{k=j}^{l-1} \frac{(-1)^k\, 2^{-\nu-2k}}{k!\,\Gamma(\nu+k+1)}\, r^{-\nu-2k-2}\,\tilde{u}(\nu+2k+2),\end{aligned}$$

thus $H_\nu^{(j)}u - H_\nu^{(k)}u = I\,\mathrm{op}_M^{1+\delta}(a_\nu)u - I\,\mathrm{op}_M^{1+\delta'}(a_\nu)u = \sum_{k=j}^{l-1} c_k(u)\,\xi^{\nu+2k}$, where the functionals c_k are given by (14). The proof is finished. $\square$

Proof of Theorems 1(c), 3. Notice that, for $u \in \mathcal{H}^{s-\delta,-1/2-\delta}(\mathbb{R}_+)$, the condition stated in Theorem 3 is equivalent to

$$\langle z\rangle^s\, a_\nu(z)\tilde{u}(z) \in L^2(\Gamma_{1+\delta}). \tag{22}$$

First let $u = H_\nu v$ for $v \in \mathcal{H}^{s,-1/2+\delta}(\mathbb{R}_+)$. Then $\tilde{u}(z) = b_\nu(z)\tilde{v}(2-z)$ for $\mathrm{Re}\,z = 1+\delta$ such that $\langle z\rangle^s\, a_\nu(z)\tilde{u}(z) = \langle z\rangle^s\,\tilde{v}(2-z) \in L^2(\Gamma_{1+\delta})$. Vice versa, if $u \in \mathcal{H}^{s-\delta,-1/2-\delta}(\mathbb{R}_+)$ and (22) is satisfied, then v defined by $\tilde{v}(z) = a_\nu(2-z)\tilde{u}(2-z)$ for $\mathrm{Re}\,z = 1-\delta$ belongs to $\mathcal{H}^{s,-1/2+\delta}(\mathbb{R}_+)$ and satisfies $u = H_\nu v$. The operator $H_\nu^{(j\to j+1)}$ is densely defined on $\mathcal{H}^{s-\delta,-1/2-\delta}(\mathbb{R}_+)$, since the condition (22) is clearly fulfilled if $\tilde{u}(z) = 0$ on $\Gamma_{1+\delta}$ close to $z = \nu+2j+2$. $\square$

Remark 2. From the above it is seen that, for δ, $\delta' \in \mathbb{R}$, $\delta > -1-\nu$, and $\delta' \le \delta$, we have

$$\begin{aligned}H_\nu\colon \mathcal{H}^{-\infty,-1/2+\delta}(\mathbb{R}_+) \cap \mathcal{H}^{-\infty,-1/2+\delta'}(\mathbb{R}_+) &\to \\ \to \mathcal{H}^{-\infty,-1/2-\delta}(\mathbb{R}_+) \cap [\omega]\,\mathcal{K}_{Q_\nu}^{-\infty,-1/2-\delta}&(\mathbb{R}_+)_{[\delta'-\delta,0]} \subseteq \mathcal{H}^{-\infty,-1/2-\delta}(\mathbb{R}_+),\end{aligned}$$

where $Q_\nu = \{(-\nu-2j,0);\, j \in \mathbb{N}\}$, cf. Lemma 5. Hence, by intersecting $\mathcal{H}^{-\infty,-1/2+\delta}(\mathbb{R}_+)$ with $\mathcal{H}^{-\infty,-1/2+\delta'}(\mathbb{R}_+)$, up to asymptotic terms of type Q_ν the behaviour near $\xi = 0$ in the image of the Hankel transformation is improved up to the conormal order $-1/2-\delta'$. Notice that $[\omega]\,\mathcal{K}_{Q_\nu}^{-\infty,-1/2-\delta}(\mathbb{R}_+)_{[\delta'-\delta,0]} = [\omega]\,\mathcal{K}^{-\infty,-1/2-\delta'}(\mathbb{R}_+)$ if $\delta' > -1-\nu$.

4. Further results

We discuss and extend results of A. H. Zemanian from [17].

4.1. Space of test functions. Zemanian proposed in [17] as space of test functions for the Hankel transformation the space $\mathcal{Z}_\nu(\mathbb{R}_+)$, defined as the space of all $u \in C^\infty(\mathbb{R}_+)$ for which, for all $j,\, l \in \mathbb{Z}_+$, the seminorm

$$u \mapsto \sup_{r\in\mathbb{R}_+} \left| r^j \Big(r^{-1}\frac{d}{dr}\Big)^l \big(r^{-\nu}u(r)\big)\right|$$

is finite. $\mathcal{Z}_\nu(\mathbb{R}_+)$ equipped with this fundamental system of seminorms is a Fréchet space. It was shown in [17] that H_ν induces a topological isomorphism from $\mathcal{Z}_\nu(\mathbb{R}_+)$ onto itself.

Recall that $\mathcal{S}(\mathbb{R}_+)$ is the space of all restrictions of functions in $\mathcal{S}(\mathbb{R})$ to $\mathbb{R}_+$, while $\mathcal{S}_0(\overline{\mathbb{R}}_+)$ is the space of all functions in $\mathcal{S}(\mathbb{R})$ supported in $\overline{\mathbb{R}}_+$.

Lemma 5. *For $\nu \in \mathbb{R}$, we have*

$$\mathcal{Z}_\nu(\mathbb{R}_+) = \big\{r^\nu\phi(r^2);\, \phi \in \mathcal{S}(\mathbb{R}_+)\big\}. \tag{23}$$

In particular, for $\nu > 0$,

$$\mathcal{Z}_\nu(\mathbb{R}_+) = \mathcal{S}^{1/2}_{Q_\nu}(\mathbb{R}_+), \tag{24}$$

with Q_ν being the asymptotic type $\{(-\nu-2j,0)\}_{j\in\mathbb{N}}$.

Proof. It is easy to verify that all functions of the form

$$r^\nu\phi_0(r^2) + \phi_1(r),$$

where $\phi_0 \in \mathcal{S}(\mathbb{R}_+)$, $\phi_1 \in \mathcal{S}_0(\overline{\mathbb{R}}_+)$, belong to $\mathcal{Z}_\nu(\mathbb{R}_+)$. Vice versa, that all functions in $\mathcal{Z}_\nu(\mathbb{R}_+)$ are of this form is stated in a somewhat different form in [17, Lemma 5.2-1]. Now the mapping

$$\phi(r) \mapsto r^\nu\phi(r^2)$$

realizes an isomorphism from $\mathcal{S}_0(\overline{\mathbb{R}}_+)$ onto $\mathcal{S}_0(\overline{\mathbb{R}}_+)$. Hence (23) follows. An immediate consequence is that, for any $\gamma < 1/2 + \nu$, it holds that $\mathcal{Z}_\nu(\mathbb{R}_+) = \mathcal{S}^\gamma_{Q_\nu}(\mathbb{R}_+)$. □

Lemma 6. *Let $\nu \in \mathbb{R}$, $\nu > 0$. Further let $u \in \mathcal{H}^{-\infty,1/2}(\mathbb{R}_+)$. Then $u \in \mathcal{S}^{1/2}_{Q_\nu}(\mathbb{R}_+)$ if and only if its Mellin transform $\tilde{u}(z)$ is meromorphic on $\mathbb{C}$ with at most simple poles at $-\nu - 2j$, $j \in \mathbb{N}$, and additionally, for every excision function $\chi \in C^\infty(\mathbb{C})$ for these poles, we have that*

$$\chi(z)\tilde{u}(z) \in \mathcal{S}(\Gamma_\beta)$$

holds uniformly for $\beta \in [\beta_0, \beta_1]$, β_0, $\beta_1 \in \mathbb{R}$, $\beta_0 < \beta_1$, where $\mathcal{S}(\Gamma_\beta)$ is the Schwartz space on the weight line Γ_β.

Proof. This kind of characterization of the Mellin image of $\mathcal{S}^\gamma_P(\mathbb{R}_+)$ for $\gamma \in \mathbb{R}$, $P \in \underline{\mathrm{As}}^{\gamma,\bullet}$ can be found, e.g., in [9, 1.2.1]. □

The characterization of the Mellin transforms of functions in $\mathcal{Z}_\nu(\mathbb{R}_+)$, together with the description of H_ν as $I\,\mathrm{op}^1_M(a_\nu)$, reproves the result that H_ν is a topological isomorphism from $\mathcal{Z}_\nu(\mathbb{R}_+)$ onto itself: for $u \in \mathcal{Z}_\nu(\mathbb{R}_+)$, $a_\nu(z)\tilde{u}(z)$ is meromorphic on $\mathbb{C}$ with at most simple poles at $\nu + 2j + 2$, $j \in \mathbb{N}$. Then, in the Mellin image, the application of I brings these poles to the places $-\nu - 2j$, $j \in \mathbb{N}$, as required. Moreover, H_ν is self-inverse on $\mathcal{Z}_\nu(\mathbb{R}_+)$, since $\mathcal{S}^{1/2}_{Q_\nu}(\mathbb{R}_+) \subset \mathcal{H}^{\infty,-1/2}(\mathbb{R}_+)$, and H_ν is self-inverse on $\mathcal{H}^{\infty,-1/2}(\mathbb{R}_+)$. The above consideration also reveals that $\mathcal{Z}_\nu(\mathbb{R}_+)$ is the smallest space containing $C^\infty_0(\mathbb{R}_+)$ having the property that H_ν is an topological isomorphism on it.

4.2. **Hankel transformation of arbitrary order.** It can be checked that the relation

$$H_{\nu+1}N_{\nu+1} = \xi H_\nu, \tag{25}$$

where $N_{\nu+1} = r^{-1}\big(-rd/dr + \nu\big)$, holds on $\mathcal{H}^{-\infty,-1/2+\delta}(\mathbb{R}_+)$, $\delta > -1 - \nu$. This was used in [17] to define the Hankel transformation H_ν on $\mathcal{Z}_\nu(\mathbb{R}_+)$ for arbitrary $\nu \in \mathbb{R}$. The considerations there were particularly based on the fact that $N_{\nu+1}$ induces an isomorphism from $\mathcal{Z}_\nu(\mathbb{R}_+)$ onto $\mathcal{Z}_{\nu+1}(\mathbb{R}_+)$.

The observation that $H_\nu = I\,\mathrm{op}^1_M(a_\nu)$ holds provides another possibility to extend to orders $\nu < 0$. The definition of the operator $I\,\mathrm{op}^{1-\delta}_M(a_\nu)$ makes sense on $\mathcal{H}^{-\infty,-1/2+\delta}(\mathbb{R}_+)$ for arbitrary

$\nu \in \mathbb{R}$ provided that $\delta > -1-\nu$. Theorem 1 continues to hold, when again denoting the operator obtained by H_ν. For instance, H_ν is an isometry on $\mathcal{H}^{s,-1/2}(\mathbb{R}_+)$ for $\nu > -1$. The corresponding variants of Theorem 2 and Theorem 3 are valid. Thus we also get modified Hankel transformations $H_\nu^{(j)}$ for $j \in \mathbb{N}$, $j \geq 1$.

This situation is pictured in Figure 1, where the axis are labelled with the order of the Hankel transformation ν and the parameter δ in (12). Thus the ν-axis corresponds to the conormal order $-1/2$ for which the isometric mapping property holds. The picture is symmetric with respect to the ν-axis in the sense that reflection yields the *inverse operator*. The grey region indicates where the Hankel transformation was initially defined, while the hatched region exemplary specifies the region in which we have $H_\nu^{(1)}$. The lines $\pm\delta = 1+\nu+2l$, $l \in \mathbb{N}$, cut out the intervals $I_j(\nu)$ from the vertical line drawn at width ν. In particular, for $-1-2l < \nu < 1-2l$, $l \in \mathbb{N}$, $l \geq 1$, we only have the intervals $I_j(\nu)$ with $|j| \geq l$, where $I_j(\nu) = (-1-\nu-2l, 1+\nu+2l)$ for $|j| = l$. Therefore, for $-1-2l < \nu < 1-2l$, $l \in \mathbb{N}$, $l \geq 1$, and $\delta \in I_l(\nu)$, $H_\nu^{(l)}$ is a bijection from $\mathcal{H}^{s,-1/2+\delta}(\mathbb{R}_+)$ onto $\mathcal{H}^{s-\delta,-1/2-\delta}(\mathbb{R}_+)$, with inverse $H_\nu^{(l)}$, especially, $H_\nu^{(l)}$ is a self-inverse isometry on $\mathcal{H}^{s,-1/2}(\mathbb{R}_+)$ in case $\delta = 0$.

Especially interesting are the points of intersection of the lines $\pm\delta = 1+\nu+2l$, $l \in \mathbb{N}$. The lines $\delta = -1-\nu-2l$ correspond to the poles of $a_\nu(z)$ at $z = \nu+2l+2$, whereas the dashed lines $\delta = 1+\nu+2l$ correspond to the zeros of $a_\nu(z)$ at $z = -\nu-2l$. For $\nu = -1-2j$, $j \in \mathbb{N}$, we see that, at $z = \nu+2l+2$ for $l \in \mathbb{N}$, $l \leq j$, the poles of the numerator of $a_\nu(z)$ are cancelled by the poles of the denominator of $a_\nu(z)$. Equally the zeros of $a_\nu(z)$ at $z = -\nu-2l$ for $l \in \mathbb{N}$, $l \leq j$, disappear. Therefore, $H_\nu = H_\nu^{(l)}$ for $\nu = -1-2j$, $l \in \mathbb{N}$, $l \leq j$. Afterwards this in part justifies the notation H_ν used in [17] to denote the extension of the Hankel transformation to $\mathcal{Z}_\nu(\mathbb{R}_+)$ for negative values of ν.

5. Application to Waves in Infinite Wedges

In this final section we consider an application to wave equations on infinite wedges. More precisely, we discuss the model equation in 1+1 dimensions

$$\partial_t^2 u - r^{-2}\big(r\partial_r\big)^2 u + r^{-2}\nu^2 u = f \text{ on } \mathbb{R}\times\mathbb{R}_+, \tag{26}$$

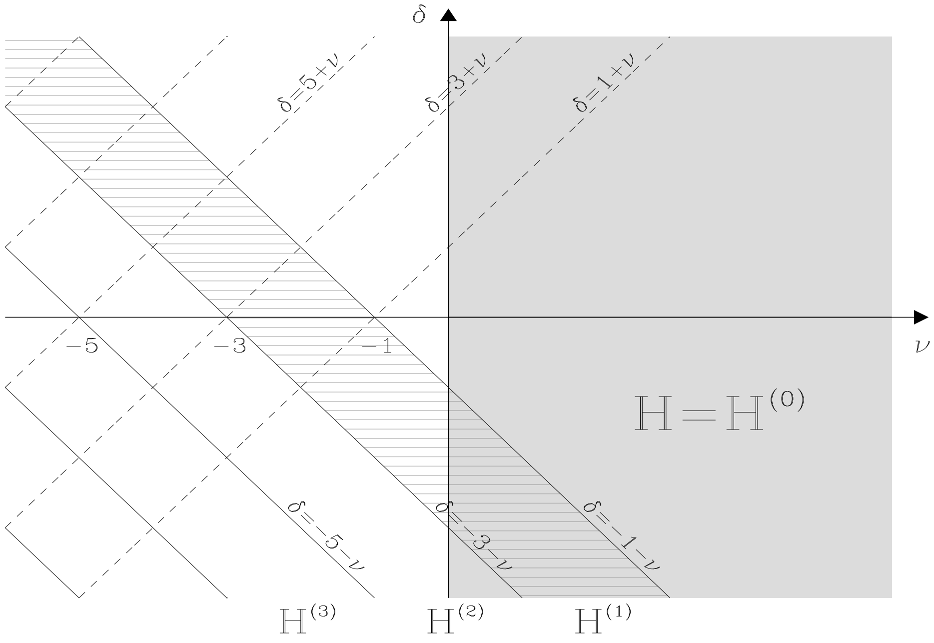

FIGURE 1. Range of parameters for the Hankel transformation

assuming that the right-hand side f fulfils $f = 0$ for $t < 0$. Throughout this section, it is thereby supposed that $\nu > 0$. Geometrically, the degeneracy at $r = 0$ in Eq. (26) corresponds to a conical point when t is fixed. Then an edge emanating from this conical point arises in time. It is not difficult to verify that (26) possesses a unique solution $u \in \bigcap_{j=0}^{s} C^j(\mathbb{R}; \mathcal{K}^{s-j+1,1/2}(\mathbb{R}_+)) \cap C^{s+1}(\mathbb{R}; \mathcal{K}^{0,-1/2}(\mathbb{R}_+))$ satisfying $u = 0$ for $t < 0$ provided that $f \in \bigcap_{j=0}^{s+1} C^j(\mathbb{R}; \mathcal{K}^{s-j,-3/2}(\mathbb{R}_+))$ for a certain $s \in \mathbb{N}$, $s \geq 1$. See, e.g., [16]. But the general expectation from the elliptic theory, cf. [9], [11], is that (26) exhibits a solution $u \in \mathcal{W}_\eta^{s+1,1/2}(\mathbb{R} \times \mathbb{R}_+)$, for $\eta > 0$, if $f \in \mathcal{W}_\eta^{s,-3/2}(\mathbb{R} \times \mathbb{R}_+)$, and s is any given real number. This section is devoted to the proof of that fact. Further some of its generalizations concerning regularity of solutions and discrete asymptotics are mentioned. Note that it is especially stated in Remark 7 that $u \in \mathcal{W}_\eta^{s+1,1/2}(\mathbb{R} \times \mathbb{R}_+)$ belongs to the space $H_\eta^{s+1}(\mathbb{R} \times \mathbb{R}_+)$ away from the edge $\{(t, r) \in \mathbb{R} \times \overline{\mathbb{R}}_+; r = 0\}$.

The Cauchy problem for Eq. (26) in which initial data are prescribed at $t = 0$ and the solution is sought for $t > 0$ is dealt with in a forthcoming paper, [1]. We emphazise that we are able to treat right-hand sides f and corresponding solutions u having variable discrete asymptotics, where even branching points may occur. Such situations arise if one considers instead of $\partial_t^2 + L_\nu$ more general differential operators which are hyperbolic in the direction of an 1-dimensional edge and possess coefficients depending on the edge parameter t.

Theorem 4. *Let s, γ, $\eta \in \mathbb{R}$, $\gamma \geq -3/2$, $\eta > 0$. Then, for any $f \in \mathcal{W}_\eta^{s,-3/2}(\mathbb{R}\times\mathbb{R}_+)$ with $f = 0$ for $t < 0$, there exists a unique solution $u \in \mathcal{W}_\eta^{s+1,1/2}(\mathbb{R}\times\mathbb{R}_+)$ to (26) satisfying $u = 0$ for $t < 0$. Moreover, $u \in \mathcal{W}_{\eta,Q_\nu}^{s+1,1/2}(\mathbb{R}\times\mathbb{R}_+)_{[-\gamma-3/2,0]}$ if $f \in \mathcal{W}_{\eta,Q_\nu}^{s,-3/2}(\mathbb{R}\times\mathbb{R}_+)_{[-\gamma-3/2,0]}$.*

Remark 3. The most interesting case occurs when $s \geq 0, \gamma = s - 3/2$. In that case we find a solution $u \in \mathcal{W}_{\eta,Q_\nu}^{s+1,-1/2}(\mathbb{R}\times\mathbb{R}_+)_{[-s-1,0]} = \mathcal{W}_{\eta,Q_\nu}^{s+1,1/2}(\mathbb{R}\times\mathbb{R}_+)_{[-s,0]}$ when $f \in \mathcal{W}_{\eta,Q_\nu}^{s,-1/2}(\mathbb{R}\times\mathbb{R}_+)_{[-s,0]} \subsetneq \mathcal{W}_{\eta,Q_\nu}^{s,-3/2}(\mathbb{R}\times\mathbb{R}_+)_{[-s,0]}$. This agrees with the situation familiar for boundary value problems, which is described in Example 3, in which we have the asymptotic type $T = \{(-j,0); j \in \mathbb{N}\}$ for Taylor asymptotics instead of Q_ν and the basic conormal order 0 instead of $-1/2$. It is further explained in [1].

Remark 4. It is not hard to see that $f \in C^\infty(\mathbb{R}; \mathcal{Z}_\nu(\mathbb{R}_+))$ with $f = 0$ for $t < 0$ leads to a unique solution $u \in C^\infty(\mathbb{R}; \mathcal{Z}_\nu(\mathbb{R}_+))$ to (26) satisfying $u = 0$ for $t < 0$. This solution is given by the formula $u(t) = \int_0^t H_\nu \xi^{-1} \sin(\xi(t-s)) H_\nu f(s)\, ds$.

As we will see in a moment in the proof of Theorem 4, the basic equation to discuss is

$$\kappa(\tau)^{-1}\hat{u}(\tau) = \langle\tau\rangle^{-2} H_\nu \frac{1}{-(\tau/\langle\tau\rangle)^2 + \xi^2} H_\nu \kappa(\tau)^{-1}\hat{f}(\tau) \tag{27}$$

which holds for $\operatorname{Im}\tau = -\eta$ and relates the given right-hand side f to the solution u. First we prove two lemmas. The kind of argumentation that applies in the proof is a certain $[\omega]$–$[1-\omega]$ *cut-off technique*. Later on a cut-off function $\omega \in C^\infty(\overline{\mathbb{R}}_+)$, i.e., it fulfils $\omega = 1$ near $r = 0$ and $\omega = 0$ for large r, shall be fixed once and for all.

We shall supply the proof of Theorem 4 for $s \geq 0$. The case $s < 0$ is analogous.

Lemma 7. *Let s, $\gamma \in \mathbb{R}$, $s \geq 0$, $\gamma \geq -3/2$. Then the Hankel transformation H_ν realizes an isomorphism*

$$\begin{aligned}
&\mathcal{K}^{s,-3/2}_{Q_\nu}(\mathbb{R}_+)_{[-\gamma-3/2,0]} \to [\omega]\,\mathcal{H}^{-1/2,-1}(\mathbb{R}_+) + \\
&\qquad + \begin{cases} [1-\omega]\left(\mathcal{H}^{s-\gamma-1/2,-\gamma-1}(\mathbb{R}_+) \cap \mathcal{H}^{-1/2,-s-1}(\mathbb{R}_+)\right) & \text{if } s \leq \gamma, \\ [1-\omega]\left(\mathcal{H}^{s-\gamma-1/2,-\gamma-1}(\mathbb{R}_+) + \mathcal{H}^{-1/2,-s-1}(\mathbb{R}_+)\right) & \text{if } -3/2 \leq \gamma \leq s. \end{cases}
\end{aligned} \tag{28}$$

Proof. First suppose that $\gamma \notin \nu + 1/2 + 2\mathbb{N}$. Then

$$\mathcal{K}^{s,-3/2}_{Q_\nu}(\mathbb{R}_+)_{[-\gamma-3/2,0]} = \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \oplus \mathcal{E}^{-3/2}_{Q_\nu}(\mathbb{R}_+)_{[-\gamma-3/2,0]}.$$

Since $\mathcal{E}^{-3/2}_{Q_\nu}(\mathbb{R}_+)_{[-\gamma-3/2,0]}$ is obviously mapped by H_ν into the right-hand side of (28), cf. Example 1, we have to treat $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$. Now

$$\mathcal{K}^{s,\gamma}(\mathbb{R}_+) = \begin{cases} \mathcal{H}^{s,\gamma}(\mathbb{R}_+) \cap \mathcal{H}^{s,s}(\mathbb{R}_+) \cap \mathcal{H}^{0,0}(\mathbb{R}_+) & \text{if } s \leq \gamma, \\ \left(\mathcal{H}^{s,\gamma}(\mathbb{R}_+) + \mathcal{H}^{s,s}(\mathbb{R}_+)\right) \cap \mathcal{H}^{0,0}(\mathbb{R}_+) & \text{if } 0 \leq \gamma \leq s, \\ \mathcal{H}^{s,\gamma}(\mathbb{R}_+) + \left(\mathcal{H}^{s,s}(\mathbb{R}_+) \cap \mathcal{H}^{0,0}(\mathbb{R}_+)\right) & \text{if } -3/2 \leq \gamma \leq 0, \end{cases}$$

for $[1-\omega]\,H^s(\mathbb{R}_+) = [1-\omega]\,(\mathcal{H}^{s,s}(\mathbb{R}_+) \cap \mathcal{H}^{0,0}(\mathbb{R}_+))$ for $s \geq 0$. Therefore, by Theorem 1 we obtain

$$\begin{aligned}
H_\nu \colon \mathcal{K}^{s,\gamma}(\mathbb{R}_+) &\to \begin{cases} \mathcal{H}^{s-\gamma-1/2,-\gamma-1} \cap \mathcal{H}^{-1/2,-s-1} \cap \mathcal{H}^{-1/2,-1} & \text{if } s \leq \gamma, \\ \left(\mathcal{H}^{s-\gamma-1/2,-\gamma-1} + \mathcal{H}^{-1/2,-s-1}\right) \cap \mathcal{H}^{-1/2,-1} & \text{if } 0 \leq \gamma \leq s, \\ \mathcal{H}^{s-\gamma-1/2,-\gamma-1} + \left(\mathcal{H}^{-1/2,-s-1} \cap \mathcal{H}^{-1/2,-1}\right) & \text{if } -3/2 \leq \gamma \leq 0 \end{cases} \\
&= [\omega]\,\mathcal{H}^{-1/2,-1} + \begin{cases} [1-\omega]\left(\mathcal{H}^{s-\gamma-1/2,-\gamma-1} \cap \mathcal{H}^{-1/2,-s-1}\right) & \text{if } s \leq \gamma, \\ [1-\omega]\left(\mathcal{H}^{s-\gamma-1/2,-\gamma-1} + \mathcal{H}^{-1/2,-s-1}\right) & \text{if } -3/2 \leq \gamma \leq s\,. \end{cases}
\end{aligned}$$

That the right-hand side of (28) is mapped by H_ν into $\mathcal{K}^{s,-3/2}_{Q_\nu}(\mathbb{R}_+)_{[-\gamma-3/2,0]}$ can be proved in the same manner. The case $\gamma \in \nu + 1/2 + 2\mathbb{N}$ follows by interpolation. $\square$

Lemma 8. *Let s, γ, $\eta \in \mathbb{R}$, $s \geq 0$, $\gamma \geq -3/2$, $\eta > 0$. Then*

$$-(\tau/\langle\tau\rangle)^2 + L_\nu \colon \mathcal{K}^{s+2,1/2}_{Q_\nu}(\mathbb{R}_+)_{[-\gamma-3/2,0]} \to \mathcal{K}^{s,-3/2}_{Q_\nu}(\mathbb{R}_+)_{[-\gamma-3/2,0]} \tag{29}$$

is a family of isomorphisms with parameter $\tau \in \mathbb{C}$, $\operatorname{Im}\tau = -\eta$, where its inverse family is given by $H_\nu\left(-(\tau/\langle\tau\rangle)^2 + \xi^2\right)^{-1} H_\nu \colon \mathcal{K}^{s,-3/2}_{Q_\nu}(\mathbb{R}_+)_{[-\gamma-3/2,0]} \to \mathcal{K}^{s+2,1/2}_{Q_\nu}(\mathbb{R}_+)_{[-\gamma-3/2,0]}$. Moreover,

$$\left\| H_\nu\left(-(\tau/\langle\tau\rangle)^2 + \xi^2\right)^{-1} H_\nu \right\|_{\mathcal{K}^{s,-3/2}(\mathbb{R}_+)_{[-\gamma-3/2,0]} \to \mathcal{K}^{s+2,1/2}_{Q_\nu}(\mathbb{R}_+)_{[-\gamma-3/2,0]}} \leq C_\eta\langle\tau\rangle \tag{30}$$

for some constant $C_\eta > 0$.

Proof. Because of $\mathcal{K}^{s+2,1/2}_{Q_\nu}(\mathbb{R}_+)_{[-\gamma-3/2,0]} = \mathcal{K}^{s+2,-3/2}_{Q_\nu}(\mathbb{R}_+)_{[-\gamma-7/2,0]}$, we have to discuss the multiplication operator

$$\begin{aligned} -(\tau/\langle\tau\rangle)^2 + \xi^2 \colon [\omega]\,\mathcal{H}^{-1/2,-1} + &\begin{cases} [1-\omega]\left(\mathcal{H}^{s-\gamma-1/2,-\gamma-3} \cap \mathcal{H}^{-1/2,-s-3}\right) & \text{if } s \leq \gamma, \\ [1-\omega]\left(\mathcal{H}^{s-\gamma-1/2,-\gamma-3} + \mathcal{H}^{-1/2,-s-3}\right) & \text{if } -3/2 \leq \gamma \leq s, \end{cases} \\ \to [\omega]\,\mathcal{H}^{-1/2,-1} + &\begin{cases} [1-\omega]\left(\mathcal{H}^{s-\gamma-1/2,-\gamma-1} \cap \mathcal{H}^{-1/2,-s-1}\right) & \text{if } s \leq \gamma, \\ [1-\omega]\left(\mathcal{H}^{s-\gamma-1/2,-\gamma-1} + \mathcal{H}^{-1/2,-s-1}\right) & \text{if } -3/2 \leq \gamma \leq s. \end{cases} \end{aligned}$$

This operator is an isomorphism, where the norm of its inverse operator is estimated by $C_\eta\langle\tau\rangle$ for some constant $C_\eta > 0$. This is true, since the norm of the operator

$$\left(-(\tau/\langle\tau\rangle)^2 + \xi^2\right)^{-1} \colon [\omega]\,\mathcal{H}^{-1/2,-1}(\mathbb{R}_+) \to [\omega]\,\mathcal{H}^{-1/2,-1}(\mathbb{R}_+)$$

is estimated by $C'_\eta\langle\tau\rangle$ for some other constant $C'_\eta > 0$. The latter follows by interpolation and duality, since the norm of $(-(\tau/\langle\tau\rangle)^2 + \xi^2)^{-1} \colon [\omega]\,\mathcal{H}^{\sigma,-1/2}(\mathbb{R}_+) \to [\omega]\,\mathcal{H}^{\sigma,-1/2}(\mathbb{R}_+)$ for $\sigma = 0, 1$ is estimated by $C'_\eta\langle\tau\rangle$. The proof is finished. □

Proof of Theorem 4. We derive the energetic inequality. Let $u \in \mathcal{W}^{s+2,\gamma+2}_\eta(\mathbb{R}\times\mathbb{R}_+)$. Define f to be the right-hand side of (26). Then $f \in \mathcal{W}^{s,\gamma}_\eta(\mathbb{R}\times\mathbb{R}_+)$. Applying the Fourier transformation in t and the Hankel transformation in r to (26), we arrive at the equation

$$-\tau^2 H_\nu\hat{u}(\tau) + \xi^2 H_\nu\hat{u}(\tau) = H_\nu\hat{f}(\tau)$$

valid for $\operatorname{Im}\tau = -\eta$. Hence we obtain Eq. (27), i.e.,

$$\kappa(\tau)^{-1}\hat{u}(\tau) = \langle\tau\rangle^{-2} H_\nu \left(-(\tau/\langle\tau\rangle)^2 + \xi^2\right)^{-1} H_\nu \kappa(\tau)^{-1}\hat{f}(\tau)$$

for $\kappa_\lambda H_\nu = \lambda^{-1} H_\nu \kappa_\lambda^{-1}$ for $\lambda > 0$. This gives

$$\int_{-\infty-i\eta}^{\infty-i\eta} \langle\tau\rangle^{2s+2} \|\kappa(\tau)^{-1}\hat{u}(\tau)\|^2_{\mathcal{K}^{s+2,1/2}_{Q_\nu}(\mathbb{R}_+)_{[-\gamma-3/2,0]}}\, d\tau \leq$$
$$\leq C_\eta^2 \int_{-\infty-i\eta}^{\infty-i\eta} \langle\tau\rangle^{2s} \|\kappa(\tau)^{-1}\hat{f}(\tau)\|^2_{\mathcal{K}^{s,\gamma}(\mathbb{R}_+)}\, d\tau,$$

where $C_\eta > 0$ is the constant from (30), i.e.,

$$\|u\|_{\mathcal{W}^{s+1,1/2}_{\eta,Q_\nu}(\mathbb{R}\times\mathbb{R}_+)_{[-\gamma-3/2,0]}} \leq C_\eta \|f\|_{\mathcal{W}^{s,\gamma}_\eta(\mathbb{R}\times\mathbb{R}_+)}.$$

Now using Remark 4 and the fact that $C_0^\infty(\mathbb{R}\times\mathbb{R}_+)$ is dense in $\mathcal{W}^{s,\gamma}_\eta(\mathbb{R}\times\mathbb{R}_+)$, the proof is concluded by standard functional-analytic arguments. □

Remark 5. Besides the result in Theorem 4 a more general statement concerning regularity can be proved. Let $P' \in \underline{\mathrm{As}}^{-3/2,\bullet}$. Then there exists an asymptotic type $P \in \underline{\mathrm{As}}^{1/2,\bullet}$ such that $f \in \mathcal{W}^{s,-3/2}_{\eta,P'}(\mathbb{R}\times\mathbb{R}_+)_{[-\gamma-3/2,0]}$ with $f = 0$ for $t < 0$, where $s, \gamma \in \mathbb{R}$, $\gamma \geq -3/2$, implies that the unique solution $u \in \mathcal{W}^{s+1,1/2}_\eta(\mathbb{R}\times\mathbb{R}_+)$ to (26) satisfying $u = 0$ for $t < 0$ belongs to $u \in \mathcal{W}^{s+1,1/2}_{\eta,P}(\mathbb{R}\times\mathbb{R}_+)_{[-\gamma-3/2,0]}$. The asymptotic type P can be explicitely calculated in terms of P'. In connection with Lemma 8, Theorem 4 says that $P = Q_\nu$ when $P' = Q_\nu$. Actually, the asymptotic type Q_ν is minimal for Eq. (26) in the sense that P found from P' is always larger or equal to Q_ν, in the natural ordering of asymptotic types.

Remark 6. There is still a concept for variable discrete asymptotics in which the asymptotic type is allowed to vary with the edge parameter t. Regarding that when a concrete differential operator is considered the singular exponents p_j in (35) are found by solving nonlinear eigenvalue problems we see that these singular exponents may exhibit a very complicated branching behaviour in t. The difficulties involved in defining variable discrete asymptotics with possible branching of singular exponents are overcame in Schulze's paper [10]. Then the foregoing remark remains true if P, P' are allowed to be asymptotic types for variable discrete asymptotics.

Appendix A.

In this appendix, we collect facts concerning cone Sobolev spaces, Mellin operators, conormal discrete asymptotics, and wedge Sobolev spaces. For further details and the general ideas behind, see [9], [11]. In the notation of asymptotics we follow [9], what is well-adapted in our situation, but slightly different from [11], where half-open weight intervals were considered.

A.1. **Cone Sobolev spaces and Mellin operators.** For given $s \in \mathbb{N}$, $\gamma \in \mathbb{R}$, the Hilbert space $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ is defined to consist of all functions u on $\mathbb{R}_+$ satisfying $r^{j-\gamma} d^j u/dr^j \in L^2(\mathbb{R}_+)$ for all $j \in \mathbb{N}$, $j \leq s$. For arbitrary $s, \gamma \in \mathbb{R}$, the Hilbert space $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ is then defined by duality and interpolation. In particular, $r^\delta \mathcal{H}^{s,\gamma}(\mathbb{R}_+) = \mathcal{H}^{s,\gamma+\delta}(\mathbb{R}_+)$.

It turns out that the spaces $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ can be introduced using the Mellin transformation,

$$Mu(z) = \tilde{u}(z) = \int_0^\infty r^{z-1} u(r)\, dr \tag{31}$$

for $u \in C_0^\infty(\mathbb{R}_+)$. It can be shown that the Mellin transformation extends by continuity to an isomorphism $M\colon r^\gamma L^2(\mathbb{R}_+) \to L^2(\Gamma_{1/2-\gamma})$, where $\Gamma_{1/2-\gamma}$ denotes the *weight line* $\{z \in \mathbb{C};\ \operatorname{Re} z = 1/2-\gamma\}$ and $L^2(\Gamma_{1/2-\gamma})$ is explained via identification of $\Gamma_{1/2-\gamma}$ with $\mathbb{R}$. More generally, an equivalent norm on $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ is given by

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+)} = \left\{ \frac{1}{2\pi i} \int_{\Gamma_{1/2-\gamma}} (1+|z|^2)^s \, |Mu(z)|^2 \, dz \right\}^{1/2}. \tag{32}$$

Henceforth, the expression $(1+|z|^2)^{1/2}$ shall be abbreviated as $\langle z \rangle$. Speaking of $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ as a Hilbert space, we always have the norm (32) in mind. For further reference, we also introduce

$$\mathcal{H}^{\infty,\gamma}(\mathbb{R}_+) = \bigcap_{s\in\mathbb{R}} \mathcal{H}^{s,\gamma}(\mathbb{R}_+), \;\; \mathcal{H}^{-\infty,\gamma}(\mathbb{R}_+) = \bigcup_{s\in\mathbb{R}} \mathcal{H}^{s,\gamma}(\mathbb{R}_+).$$

Given a function $h(z)$ on $\Gamma_{1/2-\gamma}$ satisfying the estimate $|h(z)| \leq C \langle z \rangle^\mu$ for certain constants C, $\mu \in \mathbb{R}$, a so-called *Mellin symbol*, we associate with h the Mellin operator $\operatorname{op}_M^{1/2-\gamma}(h)$ by the convention

$$\operatorname{op}_M^{1/2-\gamma}(h)u(r) = \frac{1}{2\pi i} \int_{\Gamma_{1/2-\gamma}} \int_0^\infty \left(\frac{r}{r'}\right)^{-z} h(z) u(r') \, \frac{dr'}{r'} dz$$

for $u \in C_0^\infty(\mathbb{R}_+)$, i.e., formally we have $\operatorname{op}_M^{1/2-\gamma}(h) = M^{-1} h(z) M$, with the Mellin transformation and its inverse being related to the weight line $\Gamma_{1/2-\gamma}$. Obviously, $\operatorname{op}_M^{1/2-\gamma}(h)$ extends by continuity to an operator

$$\mathrm{op}_M^{1/2-\gamma}(h)\colon \mathcal{H}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{H}^{s-\mu,\gamma}(\mathbb{R}_+). \tag{33}$$

If $h \in S^\mu(\Gamma_{1/2-\gamma})$ is a pseudo-differential symbol on $\Gamma_{1/2-\gamma}$, then $\mathrm{op}_M^{1/2-\gamma}(h)$ becomes a pseudo-differential operator of order μ on $\mathbb{R}_+$.

We still need a variant of the cone Sobolev spaces that is defined by

$$\mathcal{K}^{s,\gamma}(\mathbb{R}_+) = [\omega]\,\mathcal{H}^{s,\gamma}(\mathbb{R}_+) + [1-\omega]\,H^s(\mathbb{R}_+),$$

where $s \in \mathbb{R}\cup\{\pm\infty\}$, $\gamma \in \mathbb{R}$. ω is a cut-off function, i.e., $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ with $\omega(r) = 1$ close to $r = 0$, fixed once and for all. The notation $[\omega]\,\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ stands for the closure of $\omega(r)\,\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ in $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$, similarly for $[1-\omega]\,H^s(\mathbb{R})$. The parameter γ is interpreted as *conormal order* of distributions near $r = 0$. Especially, $\mathcal{K}^{0,0}(\mathbb{R}_+) = L^2(\mathbb{R}_+)$.

The multiplicative group $\mathbb{R}_+$ acts on $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ as a strongly continuous group $\{\kappa_\lambda\}_{\lambda>0}$ by

$$\kappa_\lambda u(r) = \lambda^{1/2}\,u(\lambda r). \tag{34}$$

$\mathbb{R}_+$ acts as a group of isometries on $L^2(\mathbb{R}_+)$.

A.2. **Conormal discrete asymptotics.** Next we introduce conormal discrete asymptotics for distributions in $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$. These asymptotics are of the form

$$u(r) \sim \sum_{j=1}^{N}\sum_{k=0}^{m_j} \omega(r) r^{-p_j} (\log r)^k\, c_{jk} \ \text{ as } r \to 0^+ \tag{35}$$

for $N \in \mathbb{N}\cup\{\infty\}$, $m_j \in \mathbb{N}$, and certain $c_{jk} \in \mathbb{C}$. Notice that, for $p \in \mathbb{C}$, $k \in \mathbb{N}$, we have $\omega(r)\,r^{-p}(\log r)^k \in \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ if and only if $\operatorname{Re} p < 1/2-\gamma$. An asymptotic type P is represented by a sequence $\{(p_j, m_j)\}_{j=1}^N \subset \mathbb{C}\times\mathbb{N}$ having the property that $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$ in case $N = \infty$. In addition some conormal order γ will be fixed. Then it is required that $\operatorname{Re} p_j < 1/2-\gamma$ holds for all j, $1 \le j \le N$. The set of all such asymptotic types is denoted by $\underline{\mathrm{As}}^{\gamma,\bullet}$. (For there are also continuous asymptotics, $^\bullet$ always indicates discrete asymptotics.) We write $\pi_{\mathbb{C}}(P) = \bigcup_{j=1}^N \{p_j\}$. Removing from a sequence representing P all terms (p_j, k) for which there is a term (p_j, m_j) in P such that $k < m_j$, we arrive at a *minimal representation* for P. The empty asymptotic type $\mathcal{O}$ is characterized by $\pi_{\mathbb{C}}(\mathcal{O}) = \emptyset$.

By a *weight interval* for asymptotics as $r \to 0^+$ we mean a closed interval $\Delta = [\vartheta, 0]$ with $-\infty < \vartheta \le 0$, or $\Delta = (-\infty, 0]$. To any such interval we associate the strip $S^\gamma_\Delta = \{z \in \mathbb{C};\ \operatorname{Re} z \in -1/2 + \gamma + \Delta\}$ in the complex Mellin plane. For $P \in \underline{\mathrm{As}}^{\gamma,\bullet}$ and Δ being finite, we define the subspace $\mathcal{K}^{s,\gamma}_P(\mathbb{R}_+)_\Delta$ of distributions in $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ obeying discrete asymptotics of type P over the weight interval Δ as follows: u in $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ belongs to $\mathcal{K}^{s,\gamma}_P(\mathbb{R}_+)_\Delta$ if $h(z) = M_{r\to z}\{\omega u\}$ fulfils

(a) $h(z) \in \mathcal{A}(\operatorname{int} S^\gamma_\Delta \setminus \pi_{\mathbb{C}}(P))$;

(b) for each $\chi(z) \in C^\infty(S^\gamma_\Delta)$ such that $\partial^\alpha_z \chi(p_j) = 0$ for all $\alpha \in \mathbb{N}^2$, $|\alpha| \le m_j$, when (p_j, m_j) is in P, and $\chi(z) = 1$ for large $|z|$ we have

$$\sup_{\beta \in \operatorname{int}\Delta} \left\| \langle z\rangle^s \chi(z) h(z) \right\|_{L^2(\Gamma_{1/2-\gamma+\beta})} < \infty. \tag{36}$$

From (a), (b) it follows in particular that $h(z)$ is meromorphic on $\operatorname{int} S^\gamma_\Delta$, with a pole at $p_j \in \operatorname{int} S^\gamma_\Delta$ of order at most $m_j + 1$, when (p_j, m_j) is in a minimal representation of P.

In case $\Delta = [\vartheta, 0]$, $-\infty < \vartheta \le 0$, and $\pi_{\mathbb{C}}(P) \cap \Gamma_{1/2-\gamma+\vartheta} = \emptyset$, the structure of $\mathcal{K}^{s,\gamma}_P(\mathbb{R}_+)_\Delta$ can be further specified. In that case let $\mathcal{E}^\gamma_P(\mathbb{R}_+)_\Delta$ denote the linear span of all functions $\omega(r) r^{-p_j} (\log r)^k$, with (p_j, m_j) being term of P, $\operatorname{Re} p_j > 1/2 - \gamma - \vartheta$, and $0 \le k \le m_j$. Then we have

$$\mathcal{K}^{s,\gamma}_P(\mathbb{R}_+)_\Delta = \mathcal{K}^{s,\gamma-\vartheta}(\mathbb{R}_+) \oplus \mathcal{E}^\gamma_P(\mathbb{R}_+)_\Delta.$$

Notice that $\mathcal{K}^{s,\gamma}_P(\mathbb{R}_+)_{[0,0]} = \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$. We put, moreover, $\mathcal{K}^{s,\gamma}_P(\mathbb{R}_+) = \mathcal{K}^{s,\gamma}_P(\mathbb{R}_+)_{(-\infty,0]} = \bigcap_{k=0}^\infty \mathcal{K}^{s,\gamma}_P(\mathbb{R}_+)_{[-k,0]}$. We also put

$$\mathcal{S}^\gamma_P(\mathbb{R}_+) = [\omega]\, \mathcal{K}^{\infty,\gamma}_P(\mathbb{R}_+) + [1-\omega]\, \mathcal{S}(\mathbb{R}_+).$$

Here $\mathcal{S}(\mathbb{R}_+)$ is the space of restrictions of functions in $\mathcal{S}(\mathbb{R})$ to $\mathbb{R}_+$. Note the particular instances $\mathcal{S}^0_T(\mathbb{R}_+) = \mathcal{S}(\mathbb{R}_+)$, where $T = \{(-j, 0);\ j \in \mathbb{N}\}$ is Taylor asymptotics, and $\mathcal{S}^0_{\mathcal{O}}(\mathbb{R}_+) = \mathcal{S}_0(\overline{\mathbb{R}}_+)$, where $\mathcal{S}_0(\overline{\mathbb{R}}_+)$ is the space of all functions in $\mathcal{S}(\mathbb{R})$ supported in $\overline{\mathbb{R}}_+$.

Example 2. For $s \ge 0$, we have $H^s(\mathbb{R}_+) = \mathcal{K}^{s,0}_T(\mathbb{R}_+)_{[-s,0]}$ and $H^s_0(\overline{\mathbb{R}}_+) = \mathcal{K}^{s,0}_{\mathcal{O}}(\mathbb{R}_+)_{[-s,0]}$, where $H^s_0(\overline{\mathbb{R}}_+)$ is the space of all functions in $H^s(\mathbb{R})$ supported in $\overline{\mathbb{R}}_+$. This is in essence shown in [2, Appendix A].

A.3. **Wedge Sobolev spaces.** For s, γ, $\eta \in \mathbb{R}$, $\eta > 0$, we introduce the wedge Sobolev space $\mathcal{W}^{s,\gamma}_{\eta}(\mathbb{R} \times \mathbb{R}_+)$ as the completion of $C_0^{\infty}(\mathbb{R} \times \mathbb{R}_+)$ with respect to the norm

$$\|u\|_{\mathcal{W}^{s,\gamma}_{\eta}(\mathbb{R}\times\mathbb{R}_+)} = \left\{ \frac{1}{2\pi} \int_{-\infty-i\tau}^{\infty-i\tau} \langle\tau\rangle^{2s} \, \|\kappa(\tau)^{-1}\hat{u}(\tau)\|^2_{\mathcal{K}^{s,\gamma}(\mathbb{R}_+)} \, d\tau \right\}^{1/2}. \tag{37}$$

Here $\hat{u}$ denotes the Fourier transform of u, $Fu(\tau) = \hat{u}(\tau) = \int_{-\infty}^{\infty} e^{-it\tau}\, u(t)\, dt$, where $\operatorname{Im}\tau = -\eta$. Further, $\kappa(\tau) = \kappa_{\langle\tau\rangle}$, where the group action κ_λ is defined in (34). The space $\mathcal{W}^{s,\gamma}_{\eta}(\mathbb{R} \times \mathbb{R}_+)$ is a Hilbert space. It consists of all distributions $u(t, r)$ of the form

$$u(t, r) = F^{-1}_{\tau\to t}\big\{\langle\tau\rangle^{1/2}\hat{v}(\tau, r\langle\tau\rangle)\big\}, \tag{38}$$

where $v \in H^s_\eta(\mathbb{R}; \mathcal{K}^{s,\gamma}(\mathbb{R}_+))$, $\hat{v}(\tau, r) = F_{t\to\tau}\big\{v(t, r)\big\}$, and Fourier transformation and its inverse transformation are related to the line $\{\tau \in \mathbb{C}; \operatorname{Im}\tau = -\eta\}$.

Remark 7. For a distribution u on $\mathbb{R} \times \mathbb{R}_+$ with $\operatorname{supp} u \subseteq \mathbb{R} \times [a, \infty)$ for some $a > 0$, we have $u \in \mathcal{W}^{s,\gamma}_{\eta}(\mathbb{R} \times \mathbb{R}_+)$ if and only if $u \in H^s_\eta(\mathbb{R} \times \mathbb{R}_+)$, with equivalence of norms and constants only depending on a, cf. [12, Proposition 2.1.19].

For $P \in \underline{\mathrm{As}}^{\gamma,\bullet}$ and Δ a finite weight interval, we introduce the subspace $\mathcal{W}^{s,\gamma}_{\eta,P}(\mathbb{R} \times \mathbb{R}_+)_\Delta$ of distributions in $\mathcal{W}^{s,\gamma}_{\eta}(\mathbb{R} \times \mathbb{R}_+)$ obeying *constant discrete asymptotics of type* P over the weight interval Δ as the space of all distributions u having the form (38) with $v \in H^s_\eta(\mathbb{R}; \mathcal{K}^{s,\gamma}_P(\mathbb{R}_+)_\Delta)$, i.e., we replace the space $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ by its subspace $\mathcal{K}^{s,\gamma}_P(\mathbb{R}_+)_\Delta$. Thus asymptotics appearing for $u \in \mathcal{W}^{s,\gamma}_{\eta,P}(\mathbb{R} \times \mathbb{R}_+)_\Delta$ are of the form

$$u(t, r) \sim \sum_{j=1}^{N} \sum_{k=0}^{m_j} F^{-1}_{\tau\to t}\Big(\omega(r\langle\tau\rangle)\langle\tau\rangle^{1/2}(r\langle\tau\rangle)^{-p_j} \log^k(r\langle\tau\rangle)\, \hat{c}_{jk}(\tau)\Big) \text{ as } r \to 0^+ \tag{39}$$

for certain $c_{jk} \in H^s(\mathbb{R})$, and N depends on Δ. More precisely, for $\Delta = [\vartheta, 0]$, $-\infty < \vartheta \leq 0$, and $\pi_{\mathbb{C}}(P) \cap \Gamma_{1/2-\gamma+\vartheta} = \emptyset$, let $\mathcal{F}^{s,\gamma}_P(\mathbb{R} \times \mathbb{R}_+)_\Delta$ be the space of all $F^{-1}_{\tau\to t}\big(\omega(r\langle\tau\rangle)\langle\tau\rangle^{1/2} \times (r\langle\tau\rangle)^{-p_j}\log^k(r\langle\tau\rangle)\hat{c}_{jk}(\tau)\big)$, where (p_j, m_j) is in P, $\operatorname{Re} p_j > 1/2 - \gamma + \vartheta$, $0 \leq k \leq m_j$, and $c_{jk} \in H^s(\mathbb{R})$. Then

$$\mathcal{W}^{s,\gamma}_{\eta,P}(\mathbb{R} \times \mathbb{R}_+)_\Delta = \mathcal{W}^{s,\gamma-\vartheta}_{\eta}(\mathbb{R} \times \mathbb{R}_+) \oplus \mathcal{F}^{s,\gamma}_P(\mathbb{R} \times \mathbb{R}_+)_\Delta.$$

For $\Delta = (-\infty, 0]$, we set $\mathcal{W}^{s,\gamma}_{\eta,P}(\mathbb{R} \times \mathbb{R}_+) = \mathcal{W}^{s,\gamma}_{\eta,P}(\mathbb{R} \times \mathbb{R}_+)_{(-\infty,0]} = \bigcap_{k=0}^{\infty} \mathcal{W}^{s,\gamma}_{\eta,P}(\mathbb{R} \times \mathbb{R}_+)_{[-k,0]}$.

Example 3. For $s \geq 0$, replacing $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ in (37) by $H^s(\mathbb{R}_+)$ and $H_0^s(\overline{\mathbb{R}}_+)$, respectively, and utilizing the group $\{\kappa_\lambda\}_{\lambda>0}$ again, we obtain the spaces $H_\eta^s(\mathbb{R}\times\mathbb{R}_+)$ and $e^{\eta t}H_0^s(\mathbb{R}\times\overline{\mathbb{R}}_+)$.

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