Fields Institute Communications
Volume 00, 0000

# Continuous methods for solving nonlinear ill-posed problems

**Ruben G. Airapetyan**
E-mail: airapet@math.ksu.edu
Department of Mathematics
Kansas State University
Manhattan, Kansas 66506-2602, U.S.A.

**Alexander G. Ramm**
E-mail: ramm@math.ksu.edu
Department of Mathematics
Kansas State University
Manhattan, Kansas 66506-2602, U.S.A.

**Alexandra B. Smirnova**
E-mail: smirn@math.ksu.edu
Department of Mathematics
Kansas State University
Manhattan, Kansas 66506-2602, U.S.A.

**Abstract.** The goal of this paper is to develop a general approach to solution of ill-posed nonlinear problems in a Hilbert space based on continuous processes with a regularization procedure. To avoid the ill-posed inversion of the Fréchet derivative operator a regularizing one-parametric family of operators is introduced. Under certain assumptions on the regularizing family a general convergence theorem is proved. The proof is based on a lemma describing asymptotic behavior of solutions of a new nonlinear integral inequality. Then the applicability of the theorem to the continuous analogs of the Newton, Gauss-Newton and simple iteration methods is demonstrated.

## 1 Introduction

Let us consider a nonlinear operator equation

$$F(z) = 0, \quad F : H \to H, \tag{1.1}$$

in a real Hilbert space $H$ (equation (1.1) in a complex Hilbert space can be treated similarly).

1991 *Mathematics Subject Classification.* Primary: 47H17. Secondary: 65J15, 58C15.

©0000 American Mathematical Society

Assume that (1.1) is solvable (not necessarily uniquely). If the Fréchet derivative of the operator $F$ has nontrivial null-space at the solution to (1.1), then one can use the classical Newton method for solution to (1.1) only under some strong assumptions on the operator $F$ (see [**11, 6**]). Otherwise in order to construct a numerical method for solution to (1.1) one needs some regularization procedure.

In the theory of ill-posed problems many different discrete methods based on a regularization are known. Many different convergence theorems for such schemes describe the efficiency of the numerical algorithms for solving various nonlinear problems, and give existence results (see, for example, [**9, 14, 20**]). However it is quite difficult to navigate in the sea of the discrete schemes and corresponding convergence theorems. Proofs of these theorems are usually based on the contraction mapping principle and are sometimes rather complicated.

On the other hand an analysis of continuous processes is based on the investigation of the asymptotical behavior of nonlinear dynamical systems in Banach and Hilbert spaces. If a convergence theorem is proved for a continuous method, one can construct various discrete schemes generated by this continuous process. Thus construction of a discrete numerical scheme is split into two parts: construction of the continuous process and numerical integration of the corresponding nonlinear operator differential equation. Convergence theorems for regularized continuous Newton-like methods are established in [**2, 4, 18**].

The goal of this paper is to develop a general approach to continuous analogs of discrete methods and to establish fairly general convergence theorem. This approach is based on an analysis of the solution to the Cauchy problem for a nonlinear differential equation in a Hilbert space. Such an analysis was done for well-posed problems in [**1**], where it was based on a usage of an integral inequality. It is more difficult to study nonlinear ill-posed problems. In this case one has to use more complicated new integral inequality (Lemma 2.2). Based on Lemma 2.2 the general theorem establishing convergence of a regularized continuous process is proved (Theorem 2.4). Applying this theorem to the regularized Newton's and simple iteration methods (for monotone operators) and to Gauss-Newton-type methods (for non-monotone operators) convergence theorems are obtained under less restrictive conditions on the equation than the theorems known for the corresponding discrete methods. According to these theorems one can choose a regularizing operator depending on the "degree of degeneracy" of the original nonlinear problem and estimate the rate of convergence of the regularized process.

The paper is organized as follows. In Section 2 Lemma 2.2 about the solution of a new integral inequality and Theorem 2.4 about the convergence of a regularized continuous process are formulated and proved. In Section 3 this theorem is applied to continuous Newton's and simple iteration methods and in Section 4 to the Gauss-Newton-type methods. A practically interesting numerical example is considered in Section 5. A lemma about nonlinear differential inequality is proved in the W Appendix.

## 2 Regularization procedure

In the well-posed case (the Fréchet derivative $F'$ of the operator $F$ is a bijection in a neighborhood of the solution of equation (1.1)) in order to solve equation (1.1) one can use the following continuous processes:

- simple iteration method:

$$\dot{z}(t) = -F(z(t)), \quad z(0) = z_0 \in H, \tag{2.1}$$

- Newton's method:

$$\dot{z}(t) = -[F'(z(t))]^{-1}F(z(t)), \quad z(0) = z_0 \in H, \tag{2.2}$$

- Gauss-Newton's method:

$$\dot{z}(t) = -[F'^{*}(z(t))F'(z(t))]^{-1}F'^{*}(z(t))F(z(t)), \quad z(0) = z_0 \in H, \tag{2.3}$$

or some of their modifications (see [**1, 16**]).

However if $F'$ is not continuously invertible (ill-posed case) one has to replace equations (2.1) -(2.3) by the corresponding regularized equations:

- regularized simple iteration method:

$$\dot{z}(t) = -[F(z(t)) + \varepsilon(t)(z(t) - z_0)], \quad z(0) = z_0 \in H, \tag{2.4}$$

- regularized Newton's method:

$$\dot{z}(t) = -[F'(z(t)) + \varepsilon(t)I]^{-1}[F(z(t)) + \varepsilon(t)(z(t) - z_0)], \quad z(0) = z_0 \in H, \tag{2.5}$$

- regularized Gauss-Newton's methods: equation (2.6) with the function $\Phi$ defined in (4.3) or in (4.24),

with an appropriate choice of the function $\varepsilon(t)$ and the point $z_0$. Here $I$ is the identity operator.

The goal of this paper is to develop a uniform approach to such regularized methods. Let us consider the Cauchy problem:

$$\dot{z}(t) = \Phi(z(t), t), \quad z(0) = z_0 \in H, \tag{2.6}$$

with an operator $\Phi : H \times [0, \infty) \to H$. The choice of $\Phi$ yields the corresponding continuous process.

In this section a general convergence theorem (Theorem 2.4) is established. In the next two sections the convergence theorems for the processes mentioned above are derived from this general theorem. In the proof of the general theorem the technique of integral inequalities is used.

The following lemma is known. It is a version of some results concerning integral inequalities (see e.g. Theorem 22.1 in [**19**]). For convenience of the reader and to make the presentation essentially self-contained we include a proof.

**Lemma 2.1** *Let $f(t, w)$, $g(t, u)$ be continuous on region $[0, T) \times D$ ($D \subset R$, $T \le \infty$) and $f(t, w) \le g(t, u)$ if $w \le u$, $t \in (0, T)$, $w, u \in D$. Assume that $g(t, u)$ is such that the Cauchy problem*

$$\dot{u} = g(t, u), \quad u(0) = u_0, \quad u_0 \in D \tag{2.7}$$

*has a unique solution. If*

$$\dot{w} \leq f(t, w), \quad w(0) = w_0 \leq u_0, \quad w_0 \in D, \tag{2.8}$$

*then $u(t) \geq w(t)$ for all $t$ for which $u(t)$ and $w(t)$ are defined.*

**Proof** Step 1. Suppose first $f(t, w) < g(t, u)$, if $w \leq u$. Since $w_0 \leq u_0$ and $\dot{w}(0) \leq f(t, w_0) < g(t, u_0) = \dot{u}(0)$, there exists $\delta > 0$ such that $u(t) > w(t)$ on $(0, \delta]$. Assume that for some $t_1 > \delta$ one has $u(t_1) < w(t_1)$. Then for some $t_2 < t_1$ one has

$$u(t_2) = w(t_2) \quad \text{and} \quad u(t) < w(t) \quad \text{for} \quad t \in (t_2, t_1].$$

One gets

$$\dot{w}(t_2) \geq \dot{u}(t_2) = g(t, u(t_2)) > f(t, w(t_2)) \geq \dot{w}(t_2).$$

This contradiction proves that there is no point $t_2$ such that $u(t_2) = w(t_2)$.

Step 2. Now consider the case $f(t, w) \leq g(t, u)$, if $w \leq u$. Define

$$\dot{u}_n = g(t, u_n) + \varepsilon_n, \quad u_n(0) = u_0, \quad \varepsilon_n > 0, \quad n = 0, 1, ...,$$

where $\varepsilon_n$ tends monotonically to zero. Then

$$\dot{w} \leq f(t, w) \leq g(t, u) < g(t, u) + \varepsilon_n, \quad w \leq u.$$

By Step 1 $u_n(t) \geq w(t)$, $n = 0, 1, \ldots$. Fix an arbitrary compact set $[0, T_1]$, $0 < T_1 < T$.

$$u_n(t) = u_0 + \int_0^t g(\tau, u_n(\tau))d\tau + \varepsilon_n t. \tag{2.9}$$

Since $g(t, u)$ is continuous, the sequence $\{u_n\}$ is uniformly bounded and equicontinuous on $[0, T_1]$. Therefore there exists a subsequence $\{u_{n_k}\}$ which converges uniformly to a continuous function $u(t)$. By continuity of $g(t, u)$ we can pass to the limit in (2.9) and get

$$u(t) = u_0 + \int_0^t g(\tau, u(\tau))d\tau, \quad t \in [0, T_1]. \tag{2.10}$$

Since $T_1$ is arbitrary (2.10) is equivalent to the initial Cauchy problem that has a unique solution. The inequality $u_{n_k}(t) \geq w(t)$, $k = 0, 1, ...$ implies $u(t) \geq w(t)$. If the solution to the Cauchy problem (2.7) is not unique, the inequality $w(t) \leq u(t)$ holds for the maximal solution to (2.7). $\square$

Our second lemma is a key to the basic result of this section, namely to Theorem 2.4.

**Lemma 2.2** *Let $\gamma(t), \sigma(t), \beta(t) \in C[0, \infty)$. If there exists a positive function $\mu(t) \in C^1[0, \infty)$ such that*

$$0 \leq \sigma(t) \leq \frac{\mu(t)}{2}\left(\gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right), \quad \beta(t) \leq \frac{1}{2\mu(t)}\left(\gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right), \quad \mu(0)v(0) < 1, \tag{2.11}$$

*then a nonnegative solution to the following inequality*

$$\dot{v}(t) \leq -\gamma(t)v(t) + \sigma(t)v^2(t) + \beta(t). \tag{2.12}$$

*satisfies the estimate:*

$$v(t) < \frac{1}{\mu(t)}. \tag{2.13}$$

**Remark 2.3** Without loss of generality one can assume $\beta(t) \geq 0$.

In [**5**] (see also Appendix) a differential inequality $\dot{u} \leq -a(t)\psi(u(t)) + b(t)$ was studied under assumptions 1) - 3) of Lemma 6.1 of the Appendix. These assumptions alone, as we show in the Appendix, do not imply the desired conclusion (6.2). We have added assumption 4) in order to prove conclusion (6.2). In Lemma 2.2 the term $-\gamma(t)v(t) + \sigma(t)v^2(t)$ (which is analogous to some extent to the term $-a(t)\psi(u(t))$) can change sign. Our Lemma 2.2 is not covered by the result in [**5**]. In particular, in Lemma 2.2 an analog of $\psi(u)$, for the case $\gamma(t) = \sigma(t) = a(t)$, is the function $\psi(u) := u - u^2$. This function goes to $-\infty$ as $u$ goes to $+\infty$, so it does not satisfy the positivity condition imposed in [**5**].

Unlike in the case of Bihari integral inequality ([**8**]) one cannot separate variables in the right hand side of inequality (2.12) and estimate $v(t)$ by a solution of the Cauchy problem for a differential equation with separating variables. The proof below is based on a special choice of the solution to the Riccati equation majorizing a solution of integral inequality (2.12).

**Proof of Lemma 2.2** Denote:

$$w(t) := v(t)e^{\int_0^t \gamma(s)ds}, \tag{2.14}$$

then (2.12) implies:

$$\dot{w}(t) \leq a(t)w^2(t) + b(t), \quad w(0) = v(0), \tag{2.15}$$

where

$$a(t) = \sigma(t)e^{-\int_0^t \gamma(s)ds}, \quad b(t) = \beta(t)e^{\int_0^t \gamma(s)ds}.$$

Consider Riccati's equation:

$$\dot{u}(t) = \frac{\dot{f}(t)}{g(t)}u^2(t) - \frac{\dot{g}(t)}{f(t)}. \tag{2.16}$$

One can check by a direct calculation that the the solution to problem (2.16) is given by the following formula [**17**, eq. 1.33]:

$$u(t) = -\frac{g(t)}{f(t)} + \left[f^2(t)\left(C - \int_0^t \frac{\dot{f}(s)}{g(s)f^2(s)}ds\right)\right]^{-1}. \tag{2.17}$$

Define $f$ and $g$ as follows:

$$f(t) := \mu^{\frac{1}{2}}(t)e^{-\frac{1}{2}\int_0^t \gamma(s)ds}, \quad g(t) := -\mu^{-\frac{1}{2}}(t)e^{\frac{1}{2}\int_0^t \gamma(s)ds}, \tag{2.18}$$

and consider the Cauchy problem for equation (2.16) with the initial condition $u(0) = v(0)$. Then $C$ in (2.17) takes the form:

$$C = \frac{1}{\mu(0)v(0) - 1}.$$

From (2.11) one gets

$$a(t) \le \frac{\dot f(t)}{g(t)}, \quad b(t) \le -\frac{\dot g(t)}{f(t)}.$$

Since $fg = -1$ one has:

$$\int_0^t \frac{\dot f(s)}{g(s) f^2(s)} ds = -\int_0^t \frac{\dot f(s)}{f(s)} ds = \frac{1}{2}\int_0^t \left(\gamma(s) - \frac{\dot\mu(s)}{\mu(s)}\right) ds.$$

Thus

$$u(t) = \frac{e^{\int_0^t \gamma(s)ds}}{\mu(t)}\left[1 - \left(\frac{1}{1-\mu(0)v(0)} + \frac{1}{2}\int_0^t \left(\gamma(s) - \frac{\dot\mu(s)}{\mu(s)}\right) ds\right)^{-1}\right]. \tag{2.19}$$

It follows from condition (2.11) that the solution to problem (2.16) exists for all $t \in [0, \infty)$ and the following inequality holds:

$$1 > 1 - \left(\frac{1}{1-\mu(0)v(0)} + \frac{1}{2}\int_0^t \left(\gamma(s) - \frac{\dot\mu(s)}{\mu(s)}\right) ds\right)^{-1} \ge \mu(0)v(0). \tag{2.20}$$

From Lemma 2.1 and from formula (2.19) one gets:

$$v(t) e^{\int_0^t \gamma(s)ds} := w(t) \le u(t) < \frac{1}{\mu(t)} e^{\int_0^t \gamma(s)ds}, \tag{2.21}$$

and thus estimate (2.13) is proved. □

**Examples.** To illustrate conditions (2.11) of Lemma 2.2 consider the following examples of functions $\gamma$, $\sigma$, $\beta$, satisfying (2.11).

**1.** Let

$$\gamma(t) = c_1(1+t)^{\nu_1}, \quad \sigma(t) = c_2(1+t)^{\nu_2}, \quad \beta(t) = c_3(1+t)^{\nu_3}, \tag{2.22}$$

where $c_2 > 0$, $c_3 > 0$. Choose $\mu(t) := c(1+t)^{\nu}$, $c > 0$. From (2.11) one gets the following conditions

$$c_2 \le \frac{cc_1}{2}(1+t)^{\nu+\nu_1-\nu_2} - \frac{c\nu}{2}(1+t)^{\nu-1-\nu_2},$$

$$c_3 \le \frac{c_1}{2c}(1+t)^{\nu_1-\nu-\nu_3} - \frac{\nu}{2c}(1+t)^{-\nu-1-\nu_3}, \quad cv(0) < 1. \tag{2.23}$$

Thus one obtains the following conditions:

$$\nu_1 \ge -1, \quad \nu_2 - \nu_1 \le \nu \le \nu_1 - \nu_3, \tag{2.24}$$

and

$$c_1 > \nu, \quad \frac{2c_2}{c_1 - \nu} \le c \le \frac{c_1 - \nu}{2c_3}, \quad cv(0) < 1. \tag{2.25}$$

Therefore for such $\gamma$, $\sigma$, $\beta$ a function $\mu$ with the desired properties exists if

$$\nu_1 \ge -1, \quad \nu_2 + \nu_3 \le 2\nu_1, \tag{2.26}$$

and

$$c_1 > \nu_2 - \nu_1, \quad 2\sqrt{c_2 c_3} \le c_1 + \nu_1 - \nu_2, \quad 2c_2 v(0) < c_1 + \nu_1 - \nu_2. \tag{2.27}$$

In this case one can choose $\nu = \nu_2 - \nu_1$, $c = \frac{2c_2}{c_1+\nu_1-\nu_2}$. However in order to have $v(t) \to 0$ as $t \to +\infty$ (the case of interest in Theorem 2.4) one needs the following conditions:

$$\nu_1 \geq -1, \quad \nu_2 + \nu_3 \leq 2\nu_1, \quad \nu_1 > \nu_3, \tag{2.28}$$

and

$$c_1 > \nu_2 - \nu_1, \quad 2\sqrt{c_2 c_3} \leq c_1, \quad 2c_2 v(0) < c_1. \tag{2.29}$$

**2.** If

$$\gamma(t) = \gamma_0, \quad \sigma(t) = \sigma_0 e^{\nu t}, \quad \beta(t) = \beta_0 e^{-\nu t}, \quad \mu(t) = \mu_0 e^{\nu t},$$

then conditions (2.11) are satisfied if

$$0 \leq \sigma_0 \leq \frac{\mu_0}{2}(\gamma_0 - \nu), \quad \beta_0 \leq \frac{1}{2\mu_0}(\gamma_0 - \nu), \quad \mu_0 v(0) < 1.$$

**3.** If

$$\gamma(t) = \frac{1}{\sqrt{\log(t+t_0)}}, \quad \mu(t) = c\log(t+t_0),$$

then conditions (2.11) are satisfied if

$$0 \leq \sigma(t) \leq \frac{c}{2}\left(\sqrt{\log(t+t_0)} - \frac{1}{t+t_0}\right),$$

$$\beta(t) \leq \frac{1}{2c\log^2(t+t_0)}\left(\sqrt{\log(t+t_0)} - \frac{1}{t+t_0}\right), \quad v(0)c\log t_0 < 1.$$

In all considered examples $\mu(t)$ can tend to infinity as $t \to +\infty$ and provide a decay of a nonnegative solution to integral inequality (2.12) even if $\sigma(t)$ tends to infinity. Moreover in the first and the third examples $v(t)$ tends to zero as $t \to +\infty$ when $\gamma(t) \to 0$ and $\sigma(t) \to +\infty$.

**Theorem 2.4** *Let $\Phi(h,t)$ be Fréchet differentiable with respect to $h$ and satisfy the following condition:*

*there exists a differentiable function $x(t)$, $x : [0, +\infty) \to H$, such that for any $h \in H$, $t \in [0, +\infty)$*

$$(\Phi(h,t), h - x(t)) \leq \alpha(t)||h - x(t)|| - \gamma(t)||h - x(t)||^2 + \sigma(t)||h - x(t)||^3, \tag{2.30}$$

*where $\alpha(t)$ is a continuous function, $\alpha(t) \geq 0$, $\gamma(t)$ and $\sigma(t)$ satisfy conditions (2.11) of Lemma 2.2 with*

$$\beta(t) := ||\dot{x}(t)|| + \alpha(t), \quad v(0) := ||z_0 - x(0)||, \quad v(t) := ||z(t) - x(t)||, \tag{2.31}$$

*and $\mu(t)$ tends to $+\infty$ as $t \to +\infty$.*

*Then problem (2.6) has a unique solution $z(t)$ defined for all $t \in [0, \infty)$, and*

$$||z(t) - x(t)|| < \frac{1}{\mu(t)}, \quad \lim_{t \to +\infty} ||z(t) - x(t)|| = 0. \tag{2.32}$$

**Remark 2.5** One can choose the regularizing operator $\Phi(h,t)$ in (2.6) such that condition (2.30) holds in the case when $F'^*(h)F'(h)$ is not boundedly invertible (see Sect. 3).

**Proof of Theorem 2.4** Since $\Phi(h,t)$ is Fréchet differentiable with respect to $h$ there exists the solution to problem (2.6) on the maximal interval $[0,T_1)$ of the existence of the solution to (2.6). One has to show that $T_1 = +\infty$. Assume $T_1 < +\infty$. Since $H$ is a real Hilbert space one has:

$$\frac{1}{2}\frac{d}{dt}||z(t)-x(t)||^2 = (\dot z - \dot x, z(t)-x(t)) = (\Phi(z(t),t), z(t)-x(t)) - (\dot x, z(t)-x(t)). \tag{2.33}$$

Therefore from (2.30) and (2.31) one obtains

$$\frac{1}{2}\frac{d}{dt}||z(t)-x(t)||^2 \le -\gamma||z(t)-x(t)||^2 + \sigma(t)||z(t)-x(t)||^3 + \beta(t)||z(t)-x(t)||. \tag{2.34}$$

Denote

$$v(t) := ||z(t)-x(t)||.$$

From (2.34) one has:

$$v(t)\dot v(t) \le -\gamma(t)v^2(t) + \sigma(t)v^3(t) + \beta(t)v(t).$$

If $v > 0$, one gets:

$$\dot v(t) \le -\gamma(t)v(t) + \sigma(t)v^2(t) + \beta(t). \tag{2.35}$$

If $v = 0$ on some interval, then inequality (2.35) is satisfied trivially because $\beta(t) \ge 0$. Thus (2.35) holds for all $t > 0$.

By Lemma 2.2 one obtains

$$||z(t)-x(t)|| \le \frac{1}{\mu(t)}, \quad \text{for} \quad t \in [0,T_1). \tag{2.36}$$

From (2.36) one concludes that $z(t)$ does not leave the ball $B_1$ centered at $x(t)$ with radius $(\min_{t\in[0,T_1]}\mu(t))^{-1} > 0$. Since $\max_{0\le t\le T_1}||x(t)|| < \infty$, one concludes that $\sup_{0\le t<T_1}||z(t)|| < \infty$. Therefore there exists a sequence $\{t_n\} \to T_1$ such that $\{z(t_n)\}$ converges weakly to some $\tilde z$. From equation (2.6) one derives the uniform boundedness of the norm $||\dot z(t)||$ on $[0,T_1)$. Thus there exists $\lim_{t\to T_1}||z(t)-\tilde z|| = 0$. Since the conditions for the uniqueness and local solvability of the Cauchy problem for equation (2.6) with initial condition $z(T_1) = \tilde z$ are satisfied, one can continue the solution to (2.6) through $T_1$. This contradicts the assumption of maximality of $T_1$, thus $T_1 = +\infty$. Moreover, from (2.13) one gets:

$$\lim_{t\to+\infty}||z(t)-x(t)|| \le \lim_{t\to+\infty}\frac{1}{\mu(t)} = 0. \tag{2.37}$$

□

## 3 Regularized Continuous Methods for Monotone Operators

In this section we apply the regularization procedure described in Sect. 2 to solve nonlinear operator equation (1.1). Assume that $F$ is Fréchet differentiable and

$$(F'(h)\xi,\xi) \ge 0 \quad \text{for all} \quad h,\xi \in H. \tag{3.1}$$

Under this assumption the operator $F'(h) + \varepsilon(t)I$ is boundedly invertible. Define $\Phi$ as follows:

$$\Phi(h,t) := -[F'(h) + \varepsilon(t)I]^{-1}[F(h) + \varepsilon(t)(h - z_0)], \tag{3.2}$$

where $z_0 \in H$ is an initial approximation point and $\varepsilon(t)$ is some positive function on the interval $[0, \infty)$. Some restrictions on $\varepsilon(t)$ will be stated in Theorem 3.10.

An outline of the convergence proof is the following. One considers an auxiliary well-posed problem:

$$F_\varepsilon(x) := F(x) + \varepsilon(x - z_0) = 0, \quad \varepsilon > 0, \tag{3.3}$$

and shows that the difference between its solution $x(t)$ and the solution $z(t)$ to problem (2.6) tends to zero as $t \to +\infty$. On the other hand one shows that $x(t)$ converges to the exact solution $y$ of equation (1.1). Thus one proves the convergence of $z(t)$ to $y$ as $t \to +\infty$.

We recall first some definitions from nonlinear functional analysis which are used below. The most essential restrictions on the operator $F$ imposed in this section are (3.1) and $w$-continuity of $F$. In particular they imply monotonicity and hemicontinuity of $F$.

**Definition 3.1** A mapping $\varphi$ is monotone in a Hilbert space $H$ if

$$(\varphi(x_1) - \varphi(x_2), x_1 - x_2) \geq 0, \quad \forall x_1, x_2 \in H.$$

**Definition 3.2** A mapping $\varphi$ is hemicontinuous at $x_0 \in H$ if the map $t \to (\varphi(x_0 + th_1), h_2)$ is continuous in a neighborhood of $t = 0$ for any $h_1, h_1 \in H$.

**Definition 3.3** A mapping $\varphi$ is strongly monotone in a Hilbert space $H$ if there exists a constant $k > 0$ such that

$$(\varphi(x_1) - \varphi(x_2), x_1 - x_2) \geq k||x_1 - x_2||^2, \quad \forall x_1, x_2 \in H.$$

**Lemma 3.4** *If $F$ is monotone and hemicontinuous then the problem (3.3) is uniquely solvable.*

**Proof of Lemma 3.4** According to [**12**, p. 100] equation (3.3) is solvable if the operator $F_\varepsilon$ is monotone, hemicontinuous and $||F_\varepsilon(x)|| \to \infty$ as $||x|| \to \infty$. For sufficiently small $\delta > 0$ from (3.3) one has:

$$\begin{aligned}||F_\varepsilon(x)||^2 &\geq ||F(x)||^2 + \varepsilon^2||x - z_0||^2 - \frac{1}{4\delta}||F(z_0)||^2 - \delta||x - z_0||^2 \\ &\geq (\varepsilon^2 - \delta)||x - z_0||^2 - C \to +\infty\end{aligned}$$

as $||x|| \to +\infty$. Here $C$ is a constant. Since $F_\varepsilon$ is strongly monotone the solution to (3.3) is unique. Therefore Lemma 3.4 is proved. □

**Remark 3.5** The result given by Lemma 3.4 is well known and its proof is given for the convenience of the reader.

Let $\rightharpoonup$ denote weak convergence in $H$.

**Definition 3.6** We say that $F$ is $w$-continuous if $x \rightharpoonup \xi$ implies $F(x) \rightharpoonup F(\xi)$.

**Lemma 3.7** *Suppose that $F$ is $w$-continuous, all the assumptions of Lemma 3.4 are satisfied, and there exists a unique solution $y$ to (1.1). Let $x(t)$ solve (3.3) for $\varepsilon = \varepsilon(t)$, and $\varepsilon(t)$ tend to zero as $t \to +\infty$. Then*

$$\lim_{t\to+\infty} ||x(t) - y|| = 0. \tag{3.4}$$

**Proof** First let us show that $x(t)$ is bounded. Indeed, it follows from (3.3) that

$$F(x(t)) - F(y) + \varepsilon(t)(x(t) - y) = \varepsilon(t)(z_0 - y).$$

Therefore

$$(F(x(t)) - F(y), x(t) - y) + \varepsilon(t)||x(t) - y||^2 = \varepsilon(t)(z_0 - y, x(t) - y). \tag{3.5}$$

This and (3.1) imply

$$||x(t) - y|| \le ||z_0 - y||. \tag{3.6}$$

Thus there exists a sequence $\{x(t_n)\}$, $t_n \to \infty$ as $n \to \infty$, which converges weakly to some element $\tilde{y} \in H$. Let us show that $\tilde{y}$ is the (unique) solution to problem (1.1). Since $F$ is $w$-continuous, $F(x(t_n)) \rightharpoonup F(\tilde{y})$. Because of the weak lower semicontinuity of the norm in a Hilbert space one has:

$$||F(\tilde{y})|| \le \liminf_{n\to\infty} ||F(x(t_n))|| = \liminf_{n\to\infty} \varepsilon(t_n)||x(t_n) - z_0|| = 0. \tag{3.7}$$

The conclusion $F(\tilde{y}) = 0$ follows from (3.7) and can also be derived directly from (3.3) with $\varepsilon = \varepsilon(t) \to 0$ as $t \to +\infty$. By the uniqueness of the solution to equation (1.1) one concludes that $\tilde{y} = y$. Let us show that the sequence $\{x(t_n)\}$ converges strongly to $y$. Indeed, from (3.5), (3.1) and the relation $x(t_n) \rightharpoonup y$, one gets:

$$||x(t_n) - y||^2 \le (z_0 - y, x(t_n) - y) \to 0 \text{ as } n \to \infty. \tag{3.8}$$

Thus

$$\lim_{n\to\infty} ||x(t_n) - y|| = 0. \tag{3.9}$$

From (3.9) it follows by the standard argument that $x(t) \to y$ as $t \to \infty$. Lemma 3.7 is proved. □

**Lemma 3.8** *Assume that $F$ is continuously Fréchet differentiable,* $\sup_{x\in H} ||F'(x)|| \le N_1$*, and condition (3.1) holds. If $\varepsilon(t)$ is continuously differentiable, then the solution $x(t)$ to problem (3.3) with $\varepsilon = \varepsilon(t)$ is continuously differentiable in the strong sense and one has*

$$||\dot{x}(t)|| \le \frac{|\dot{\varepsilon}(t)|}{\varepsilon(t)}||y - z_0||, \quad t \in [0, +\infty). \tag{3.10}$$

**Proof** Fréchet differentiability of $F$ implies hemicontinuity of $F$. Therefore problem (3.3) with $\varepsilon = \varepsilon(t)$ is uniquely solvable. The differentiability of $x(t)$ with respect to $t$ follows from the implicit function theorem [2]. To derive (3.10) one differentiates equation (3.3) and uses the estimate $\left\|[F'(x(t))+\varepsilon(t)I]^{-1}\right\| \le \frac{1}{\varepsilon(t)}$. The result is:

$$||\dot{x}(t)|| = |\dot{\varepsilon}(t)| \cdot ||[F'(x(t))+\varepsilon(t)I]^{-1}(x(t)-z_0)|| \le \frac{|\dot{\varepsilon}(t)|}{\varepsilon(t)}||x(t)-z_0||. \tag{3.11}$$

Here we have used the estimate

$$||x(t)-z_0|| \le ||y-z_0||, \tag{3.12}$$

which can be derived from (3.3) similarly to the derivation of (3.6). Thus estimate (3.10) follows from (3.11) and (3.12). □

**Lemma 3.9** *Assume that* $\varepsilon = \varepsilon(t) > 0$, *$F$ is twice Fréchet differentiable, condition (3.1) holds, and*

$$||F'(x)|| \le N_1, \quad ||F''(x)|| \le N_2 \quad \forall x \in H. \tag{3.13}$$

*Then for the operator* $\Phi$ *defined by (3.2) and* $x(t)$, *the solution to (3.3) with* $\varepsilon = \varepsilon(t)$, *estimate (2.30) holds with*

$$\alpha(t) \equiv 0, \quad \gamma(t) \equiv 1, \text{ and } \sigma(t) := \frac{N_2}{2\varepsilon(t)}. \tag{3.14}$$

**Proof** Since $x(t)$ is the solution to (3.3) applying Taylor's formula one gets:

$$(\Phi(h,t), h-x(t)) = -([F'(h)+\varepsilon(t)I]^{-1}[F(h)-F(x(t))+\varepsilon(t)(h-x(t))], h-x(t))$$

$$\le -\left([F'(h)+\varepsilon(t)I]^{-1}[F'(h)(h-x(t))+\varepsilon(t)(h-x(t))], h-x(t)\right) + \frac{N_2||h-x(t)||^3}{2\varepsilon(t)}$$

$$= -||h-x(t)||^2 + \frac{N_2||h-x(t)||^3}{2\varepsilon(t)}. \tag{3.15}$$

From (3.15) and (2.30) the conclusion of Lemma 3.9 follows. □

Let us state the main result of this section.

**Theorem 3.10** *Assume:*

1. *problem (1.1) has a unique solution* $y$;
2. *$F$ is w-continuous, twice Fréchet differentiable and inequalities (3.1), (3.13) hold;*
3. $\varepsilon(t) > 0$ *is continuously differentiable and monotonically tends to* $0$,
$$C_\varepsilon := \max_{t\in[0,\infty)} \frac{\varepsilon(0)|\dot{\varepsilon}(t)|}{\varepsilon^2(t)} < 1; \tag{3.16}$$
4. $$\varepsilon(0) > \frac{N_2||z_0-y||}{1-C_\varepsilon}\max\left\{1, \frac{2C_\varepsilon}{1-C_\varepsilon}\right\}; \tag{3.17}$$
5. $\Phi$ *is defined by (3.2).*

*Then Cauchy problem (2.6) has a unique solution $z(t)$ for $t \in [0,+\infty)$ and*

$$||z(t)-x(t)|| \le \frac{1-C_\varepsilon}{N_2}\varepsilon(t), \quad \lim_{t\to+\infty} ||z(t)-y|| = 0. \tag{3.18}$$

**Remark 3.11** First notice that Theorem 3.10 establishes convergence for any initial approximation point $z_0$ if $\varepsilon(t)$ is appropriately chosen. To make an appropriate choice of $\varepsilon(t)$ one has to choose some function $\varepsilon(t)$ satisfying condition (3.16). Examples of such functions $\varepsilon(t)$ are given below. One can observe that condition (3.16) is invariant with respect to a multiplication $\varepsilon(t)$ by a constant. Therefore one can choose $\varepsilon(t)$ satisfying condition (3.17) by a multiplication of the original $\varepsilon(t)$ by a sufficiently large constant. If $\frac{|\dot\varepsilon(t)|}{\varepsilon^2(t)}$ is not increasing, then in condition (3.17) $C_\varepsilon := \max_{t\in[0,\infty)} \frac{\varepsilon(0)|\dot\varepsilon(t)|}{\varepsilon^2(t)}$ can be replaced by $C_\varepsilon := \frac{|\dot\varepsilon(0)|}{\varepsilon(0)}$.

**Remark 3.12** In order to get an estimate of the convergence rate for $||x(t)-y||$ one has to make some additional assumptions either on $F(x)$ or on the choice of the initial approximation $z_0$. Without such assumptions one cannot give an estimate of the convergence rate. Indeed, as a simple example consider the scalar equation $F(x) := x^m = 0$. Then one gets the following algebraic equation for $x(\varepsilon)$:

$$F_\varepsilon(x) := x^m + \varepsilon(x - z_0) = 0. \tag{3.19}$$

Assume $m$ is a positive integer and $z_0 > 0$. It is known that the solution to this equation is an algebraic function which can be represented by the Puiseux series: $x = \sum_{j=1}^{\infty} c_j \varepsilon^{\frac{j}{p}}$ in some neighborhood of zero. Thus $x = c_1\varepsilon^{\frac{1}{p}}(1+O(\varepsilon))$ as $\varepsilon \to 0$. Now from (3.19) one gets:

$$c_1^m \varepsilon^{\frac{m}{p}}(1+O(\varepsilon)) + c_1\varepsilon^{1+\frac{1}{p}}(1+O(\varepsilon)) = z_0\varepsilon.$$

Thus $p = m$, $c_1 = z_0^{\frac{1}{m}}$ and $x(\varepsilon) = z_0^{\frac{1}{m}}\varepsilon^{\frac{1}{m}}(1+O(\varepsilon))$. For $\varepsilon = 0$ one gets the solution $y = 0$. Therefore

$$|x(\varepsilon) - y| \sim \varepsilon^{\frac{1}{m}}, \quad \varepsilon \to 0, \tag{3.20}$$

where $m$ can be chosen arbitrary large.

Below in Propositions 3.13 and 3.14 some sufficient conditions are given that allow one to obtain the estimates for $||x(t)-y||$.

**Proof of Theorem 3.10** Choosing $\mu(t) = \frac{\lambda}{\varepsilon(t)}$, where $\lambda$ is a constant, from conditions (2.11) and (2.31) one gets the following inequalities:

$$N_2 \le \lambda\left(1 - \frac{|\dot\varepsilon(t)|}{\varepsilon(t)}\right), \tag{3.21}$$

$$2||z_0 - y||\frac{|\dot\varepsilon(t)|}{\varepsilon^2(t)} \le \frac{1}{\lambda}\left(1 - \frac{|\dot\varepsilon(t)|}{\varepsilon(t)}\right), \quad \lambda||z_0 - x(0)|| < \varepsilon(0). \tag{3.22}$$

Choose

$$\lambda := \frac{N_2}{1 - C_\varepsilon}. \tag{3.23}$$

It follows from (3.16) that (3.21) holds. From (3.17) one gets:

$$\varepsilon(0) > \frac{N_2 ||z_0 - y||}{1 - C_\varepsilon}. \tag{3.24}$$

On the other hand from (3.6) it follows that

$$||z_0 - x(0)|| \leq ||z_0 - y||. \tag{3.25}$$

Thus from (3.24), (3.25) and (3.23) one obtains the second inequality in (3.22). Using (3.17) once again, one gets:

$$\varepsilon(0) \geq \frac{2N_2 C_\varepsilon ||z_0 - y||}{(1 - C_\varepsilon)^2}. \tag{3.26}$$

This inequality and (3.16) imply:

$$\lambda = \frac{N_2}{1 - C_\varepsilon} \leq \frac{1 - C_\varepsilon}{2||z_0 - y||\frac{|\dot{\varepsilon}(t)|}{\varepsilon^2(t)}}. \tag{3.27}$$

By (3.16) one gets

$$\frac{N_2}{1 - C_\varepsilon} \leq \frac{1 - \frac{|\dot{\varepsilon}(t)|}{\varepsilon(t)}}{2||z_0 - y||\frac{|\dot{\varepsilon}(t)|}{\varepsilon^2(t)}}. \tag{3.28}$$

The first inequality in (3.22) is equivalent to (3.28) for $\lambda$ chosen in (3.23). Therefore one gets inequality (3.18) by applying Theorem 2.4, while the second relation (3.18) follows from (3.4), inequality (3.18) and the triangle inequality:

$$||z(t) - y|| \leq ||z(t) - x(t)|| + ||x(t) - y||.$$

Theorem 3.10 is proved. □

**Examples.**

**1.** Let $\varepsilon(t) = \varepsilon_0(t_0 + t)^{-\nu}$, $\varepsilon_0$, $t_0$ and $\nu$ are positive constants. Then $C_\varepsilon = \frac{\nu}{t_0}$ and condition (3.16) is satisfied if $\nu \in (0, 1]$ and $t_0 > \nu$.

**2.** If $\varepsilon(t) = \frac{\varepsilon_0}{\log(t_0 + t)}$, then $C_\varepsilon = \frac{1}{t_0 \log t_0}$ and condition (3.16) is satisfied if $t_0 \log t_0 > 1$.

Note that if $\varepsilon(t) = \varepsilon_0 e^{-\nu t}$ then condition (3.16) is not satisfied.

**Proposition 3.13** *Let the assumptions of Theorem 3.10 hold. Suppose also that the following inequality holds:*

$$(F(h), h - y) \geq c||h - y||^{1+a}, \quad a > 0. \tag{3.29}$$

*Then for the solution $z(t)$ to problem (2.6) the following estimate holds:*

$$||z(t) - y|| = O\left(\varepsilon^{\frac{1}{a}}(t)\right). \tag{3.30}$$

**Proof** Denote $||x(t)-y|| := \varrho(t)$. Since $F(y)=0$, inequality (3.5) implies

$$c\varrho^{1+a}(t)+\varepsilon(t)\varrho^2(t) \le \varepsilon(t)||z_0-y||\varrho(t) \tag{3.31}$$

and $\varrho(t)\to 0$ as $t\to+\infty$. This inequality can be reduced to

$$c\varrho^{a}(t)+\varepsilon(t)\varrho(t) \le \varepsilon(t)||z_0-y||. \tag{3.32}$$

Thus $\varrho^a(t) \le \frac{||z_0-y||}{c}\varepsilon(t)$, and

$$||x(t)-y|| \le \left(\frac{||z_0-y||}{c}\right)^{\frac{1}{a}} \varepsilon^{\frac{1}{a}}(t). \tag{3.33}$$

Combining this estimate with estimate (3.18) for $||z(t)-x(t)||$ one completes the proof. $\square$

**Example.** In the case of a scalar function $f(h)$ and even integer $a>0$ the estimate $f(h)(h-y)\ge c|h-y|^{1+a}$ means that $f(h)=(h-y)^a g(h)$, where $g(h)\ge c>0$, and hence $y$ is a zero of the multiplicity $a$ for $f$.

**Proposition 3.14** *Let all the assumptions of Theorem 3.10 hold and there exists $v\in H$ such that*

$$z_0-y=F'(y)v, \quad ||v||<\frac{2}{N_2}. \tag{3.34}$$

*Then for the solution $z(t)$ to problem (2.6) the following convergence rate estimate holds:*

$$||z(t)-y|| \le \left[\frac{1-C_\varepsilon}{N_2}+\frac{4||v||}{2-N_2||v||}\right]\varepsilon(t). \tag{3.35}$$

**Proof** From (3.3) for an arbitrary $\varepsilon>0$ one gets

$$F(x)-F(y)+\varepsilon(x-y)=\varepsilon(z_0-y).$$

Therefore by the Lagrange formula one has:

$$\left\{\int_0^1 (F'(y+s(x-y))ds+\varepsilon I\right\}(x-y)=\varepsilon(z_0-y). \tag{3.36}$$

Introduce the notation $Q_\varepsilon(x) := \int_0^1 (F'(y+s(x-y))ds+\varepsilon I$. From (3.36) it follows that

$$||x-y||=\varepsilon||Q_\varepsilon^{-1}(x)Q_0(y)v|| \le \varepsilon||Q_\varepsilon^{-1}(x)(Q_0(y)-Q_\varepsilon(x))v||+\varepsilon||Q_\varepsilon^{-1}(x)Q_\varepsilon(x)v||.$$

Since $Q_\varepsilon(x)=Q_0(x)+\varepsilon I$, one obtains

$$||x-y|| \le \varepsilon||Q_\varepsilon^{-1}(x)(Q_0(y)-Q_0(x))v||+\varepsilon||Q_\varepsilon^{-1}(x)\varepsilon v||+\varepsilon||v||$$

$$\le \frac{N_2}{2}||x-y||\,||v||+2\varepsilon||v||. \tag{3.37}$$

So, from (3.18), (3.34) and (3.37) for $\varepsilon=\varepsilon(t)$ and correspondingly $x=x(t)$ satisfying the assumptions of Theorem 3.10 one concludes that estimate (3.35) holds. $\square$

Now we describe the simple iteration scheme for solving nonlinear equation (1.1). Define:

$$\Phi(h,t) := -[F(h) + \varepsilon(t)(h - z_0)], \quad \varepsilon(t) > 0. \tag{3.38}$$

**Lemma 3.15** *Assume that $F$ is monotone, $\Phi$ is defined by (3.38), and $x(t)$ is a solution to problem (3.3) with $\varepsilon = \varepsilon(t) > 0$, $t \in [0, +\infty)$. Then for the positive function $\gamma(t) := \varepsilon(t)$ and for $\sigma(t) = \alpha(t) \equiv 0$ estimate (2.30) holds.*

**Proof** Since $x(t)$ is a solution to problem (3.3), by the monotonicity of $F$ one has:

$$(\Phi(h,t), h - x(t)) = -(F(h) - F(x(t), h - x(t)) - \varepsilon(t)(h - x(t), h - x(t))$$

$$\leq -\varepsilon(t)||h - x(t)||^2. \tag{3.39}$$

Lemma 3.15 is proved. □

Lemma 3.15 together with Lemma 3.16 presented below allow one to formulate the convergence result concerning the simple iteration procedure (see Theorem 3.17).

**Lemma 3.16** *Let $\nu(t)$ be integrable on $[0, +\infty)$. Suppose that there exists $T \geq 0$ such that $\nu(t) \in C^1[T, +\infty)$ and*

$$\nu(t) > 0, \quad -\frac{\dot{\nu}(t)}{\nu^2(t)} \leq C, \quad \textit{for} \quad t \in [T, +\infty). \tag{3.40}$$

*Then*

$$\lim_{t \to +\infty} \int_0^t \nu(\tau) d\tau = +\infty. \tag{3.41}$$

**Proof** One can integrate (3.40)

$$-\int_T^t \frac{\dot{\nu}(\tau)}{\nu^2(\tau)} d\tau \leq \int_T^t C d\tau, \quad t \in [T, +\infty)$$

and get

$$\frac{1}{\nu(t)} \leq C(t - T) + \frac{1}{\nu(T)}.$$

Without loss of generality we can assume that $C > 0$, and then

$$\nu(t) \geq \frac{1}{C(t - T) + \frac{1}{\nu(T)}}.$$

Integrating this inequality one gets (3.41) and completes the proof. □

Lemmas 3.4 - 3.8 and Lemmas 3.15 - 3.16 imply the following result.

**Theorem 3.17** *Assume that:*

1. *problem (1.1) has a unique solution $y$;*
2. *$F$ is w-continuous and monotone;*

3. *$F$ is continuously Fréchet differentiable and*
$$||F'(x)|| \le N_1, \quad \forall x \in H; \tag{3.42}$$
4. *$\varepsilon(t) > 0$ is continuously differentiable and tends to zero monotonically as $t \to +\infty$, and $\lim_{t\to+\infty} \frac{\dot\varepsilon(t)}{\varepsilon^2(t)} = 0$.*

*Then, for $\Phi$ defined by (3.38), Cauchy problem (2.6) has a unique solution $z(t)$ all for $t \in [0, +\infty)$ and*
$$\lim_{t\to\infty} ||z(t) - y|| = 0.$$

**Proof** In order to verify the assumptions of Theorem 2.4 we use estimate (3.39) to conclude that $\alpha(t) = \sigma(t) = 0$ and $\gamma(t) = \varepsilon(t)$ in formula (2.30). By (2.31) $\beta(t) = ||\dot x(t)||$ because $\alpha(t) = 0$. By (3.10)
$$\beta(t) = ||\dot x(t)|| \le \frac{|\dot\varepsilon(t)|}{\varepsilon(t)} ||y - z_0||.$$
To apply Theorem 2.4 one has to find a function $\mu(t) \in C^1[0, +\infty)$ satisfying (2.11) that is
$$\frac{|\dot\varepsilon(t)|}{\varepsilon(t)} ||y - z_0|| \le \frac{1}{2\mu(t)} \left( \varepsilon(t) - \frac{\dot\mu(t)}{\mu(t)} \right), \quad \mu(0)||x(0) - z_0|| < 1. \tag{3.43}$$
Such a function can be chosen as the solution to the differential equation
$$-\frac{\dot\mu(t)}{\mu^2(t)} + \frac{\varepsilon(t)}{\mu(t)} = \frac{A|\dot\varepsilon(t)|}{\varepsilon(t)}, \tag{3.44}$$
where $A := 2||y - z_0||$. Denote $\rho(t) := \frac{1}{\mu(t)}$. Then
$$\dot\rho(t) + \varepsilon(t)\rho(t) = \frac{A|\dot\varepsilon(t)|}{\varepsilon(t)}$$
and
$$\rho(t) = \left[ A \int_0^t \frac{|\dot\varepsilon(s)|}{\varepsilon(s)} e^{\int_0^s \varepsilon(\tau)d\tau} ds + \frac{1}{\mu(0)} \right] e^{-\int_0^t \varepsilon(\tau)d\tau}. \tag{3.45}$$
Since by Lemma 3.16 $e^{\int_0^t \varepsilon(\tau)d\tau} \to \infty$ as $t \to \infty$ one can apply L'Hôspital's rule to obtain from (3.45) and condition 4 that
$$\lim_{t\to+\infty} \rho(t) = \lim_{t\to+\infty} \frac{|\dot\varepsilon(t)|}{\varepsilon^2(t)} = 0.$$
Therefore $\mu(t)$ tends to $+\infty$. To complete the proof one can take $\mu(0)$ sufficiently small for the second inequality in (3.43) to hold. By Theorem 2.4 one concludes that $||z(t) - x(t)|| \to 0$ as $t \to +\infty$ and by Lemma 3.7 that $||x(t) - y|| \to 0$ as $t \to +\infty$. Therefore it follows from the estimate:
$$||z(t) - y|| \le ||z(t) - x(t)|| + ||x(t) - y||,$$
that $||z(t) - y|| \to 0$ as $t \to +\infty$. $\square$

**Remark 3.18** From the proof it is clear that one can get the estimate $||z(t) - x(t)|| \le \frac{1}{\mu(t)} \to 0$ as $t \to +\infty$, and for the term $||x(t) - y||$ one can get the rate of convergence if some additional assumptions are made on $F$ or on $z_0$ (see Propositions 3.13, 3.14, and also Remark 3.12).

**Remark 3.19** The result obtained in Theorem 3.17 is similar to the result in [**3**]. The assumptions in [**3**] are slightly different. The method of investigation in [**3**] is based on a linear differential inequality which is a particular case of (2.12) with $\sigma(t) \equiv 0$. This linear differential inequality has been used often in the literature by many authors.

**Examples.**

**1.** Let $\varepsilon(t) = \varepsilon_0(1+t)^{-\nu}$, $\varepsilon_0$ and $\nu$ are positive constants. Then the assumptions of Theorem 3.17 are satisfied if $\nu \in (0,1)$.

**2.** If $\varepsilon(t) = \frac{\varepsilon_0}{\log(1+t)}$, then the assumptions of Theorem 3.17 are satisfied.

If $\varepsilon(t) = \varepsilon_0 e^{-\nu t}$ then condition 4 of Theorem 3.17 is not satisfied.

## 4 Regularized Methods for Non-monotone Operators

In this section we discuss two approaches to the regularization of the Gauss-Newton-type schemes for nonlinear equations with non-monotone operators. To describe the first one, assume that $F$ in (1.1) is compact and Fréchet differentiable. Denote:

$$T(h) := F'^*(h)F'(h), \quad T_\varepsilon(h) := T(h) + \varepsilon I. \tag{4.1}$$

Then $T(h)$ is a nonnegative self-adjoint compact operator. Such an operator cannot be boundedly invertible if $H$ is infinite-dimensional. One has:

$$||T_\varepsilon^{-1}(h)|| \le \frac{1}{\varepsilon} \tag{4.2}$$

for any $\varepsilon > 0$. Define $\Phi$:

$$\Phi(h,t) := (P_{\varepsilon(t)}(\xi) - I)(h - z_0) - P_{\varepsilon(t)}(\xi)T_{\varepsilon(t)}^{-1}(h)F'^*(h)F(h), \quad \varepsilon(t) > 0. \tag{4.3}$$

Here $\xi \in H$ is a fixed element, which will be chosen so that inequality (4.6) (see below) holds, and

$$P_\varepsilon(\xi) := \int_\varepsilon^{N_1^2} dE(s), \tag{4.4}$$

$E(s) := E(s,\xi)$ is the resolution of the identity of the self-adjoint operator $T(\xi)$, and

$$\sup_{h \in H} ||F'(h)|| \le N_1, \tag{4.5}$$

so that $||T(\xi)|| \le N_1^2$.

**Lemma 4.1** *Assume that:*

1. *problem (1.1) has a unique solution y;*
2. *$\Phi$ is defined by (4.3), F is compact, twice Fréchet differentiable and inequalities (3.13) hold;*

3. *there exist* $\xi \in H$ *and* $\varepsilon_0 > 0$ *such that*

$$C := \sup_{\varepsilon \in (0,\varepsilon_0]} ||P_\varepsilon(\xi) T_\varepsilon^{-1}(\xi)\{T(y) - T(\xi)\}|| < \frac{1}{2}; \tag{4.6}$$

4. $0 < \varepsilon(t) \leq \varepsilon_0$.

*Then there exist positive functions* $\alpha(t)$, $\gamma(t)$ *and* $\sigma(t)$, $t \in [0, +\infty)$, *such that estimate (2.30) holds with* $x(t)$ *replaced by* $y$.

**Remark 4.2** Condition (4.6) contains a priori information about a nonlinear operator $F$. This condition allows one to get a convergence rate for ill-posed problem (1.1). It is always satisfied in a well-posed case (for a boundedly invertible operator $T$) if $\xi$ is sufficiently close to $y$. However it is not clear yet how restrictive this condition is, and how it is related to other conditions that one has to use in order to prove the convergence of the process in ill-posed cases.

**Proof of Lemma 4.1** Using the polar decomposition $F'(h) = U(F'^*(h)F'(h))^{\frac{1}{2}}$, where $U$ is a partial isometry, one gets $F'^*(h) = T^{\frac{1}{2}}U^*$, and, since $||U^*|| = 1$, one obtains:

$$||T_{\varepsilon(t)}^{-1}(h)F'^*(h)|| \leq ||T_{\varepsilon(t)}^{-1}(h)T^{\frac{1}{2}}(h)|| \leq \max_{0 \leq s < +\infty} \frac{\sqrt{s}}{s + \varepsilon(t)} = \frac{1}{2\sqrt{\varepsilon(t)}}. \tag{4.7}$$

Using (4.3) and the relation

$$F(h) = F(h) - F(y) = F'(h)(h - y) + R(y, h), \tag{4.8}$$

where $||R(y,h)|| \leq \frac{N_2}{2}||h - y||^2$, one gets

$$\begin{aligned}(\Phi(h,t), h - y) &= ((P_{\varepsilon(t)}(\xi) - I)(h - z_0), h - y) - (P_{\varepsilon(t)}(\xi)T_{\varepsilon(t)}^{-1}(h)T(h)(h - y), h - y) \\ &+ \frac{N_2}{4\sqrt{\varepsilon(t)}}||h - y||^3 \leq -||h - y||^2 + ||(P_{\varepsilon(t)}(\xi) - I)(y - z_0)|| \, ||h - y|| \\ &+ \varepsilon(t)(P_{\varepsilon(t)}(\xi)T_{\varepsilon(t)}^{-1}(h)(h - y), h - y) + \frac{N_2}{4\sqrt{\varepsilon(t)}}||h - y||^3. \end{aligned} \tag{4.9}$$

Also one has the following estimates:

$$||P_{\varepsilon(t)}(\xi)T_{\varepsilon(t)}^{-1}(\xi)|| \leq \max_{s \geq \varepsilon(t)} \frac{1}{s + \varepsilon(t)} = \frac{1}{2\varepsilon(t)}, \tag{4.10}$$

and

$$\begin{aligned}||T(h) - T(y)|| &\leq ||F'^*(h)[F'(h) - F'(y)] + [F'^*(h) - F'^*(y)]F'(y)|| \\ &\leq 2N_1N_2||h - y||. \end{aligned} \tag{4.11}$$

Thus, using the identity $A^{-1} - B^{-1} = -B^{-1}(A - B)A^{-1}$ and inequalities (4.10), (4.11), one obtains:

$$\begin{aligned}\varepsilon(t)(P_{\varepsilon(t)}(\xi)T_{\varepsilon(t)}^{-1}(h)(h - y), h - y) &= \varepsilon(t)(P_{\varepsilon(t)}(\xi)T_{\varepsilon(t)}^{-1}(\xi)(h - y), h - y) \\ &+ \varepsilon(t)(P_{\varepsilon(t)}(\xi)\{T_{\varepsilon(t)}^{-1}(h) - T_{\varepsilon(t)}^{-1}(\xi)\}(h - y), h - y) \leq \frac{1}{2}||h - y||^2 \end{aligned}$$

$$+||P_{\varepsilon(t)}(\xi)T^{-1}_{\varepsilon(t)}(\xi)||\,||T(h)-T(y)||\,||h-y||^2+||P_{\varepsilon(t)}(\xi)T^{-1}_{\varepsilon(t)}(\xi)\{T(y)-T(\xi)\}||\,||h-y||^2$$

$$\leq \left(\frac{1}{2}+C\right)||h-y||^2+\frac{N_1N_2}{\varepsilon(t)}||h-y||^3. \tag{4.12}$$

Define:

$$\gamma(t)\equiv\gamma := \frac{1}{2}-C; \tag{4.13}$$

$$\sigma(t) := \frac{N_1N_2}{\varepsilon(t)}+\frac{N_2}{4\sqrt{\varepsilon(t)}}; \tag{4.14}$$

$$\alpha(t) := ||(P_{\varepsilon(t)}(\xi)-I)(y-z_0)||. \tag{4.15}$$

These functions are positive if (4.6) holds. If $\epsilon(t)\to 0$ then $\sigma(t)\to+\infty$ and $\alpha(t)\to 0$ as $t\to+\infty$. Comparing (4.9) - (4.15) with (2.30) and applying Lemma 2.2 one completes the proof. □

**Theorem 4.3** *Suppose that the assumptions of Lemma 4.1 are satisfied and:*

1. $P(y-z_0)=0$, *where $P$ is an orthonormal projector onto the null-space of $T(\xi)$;*
2. $\varepsilon(t)>0$ *is continuously differentiable, monotonically tends to* $0$, *and*

$$C_0 := \min_{t\in[0,+\infty)}\left\{\gamma-\frac{|\dot\varepsilon(t)|}{\varepsilon(t)}\right\}>0, \tag{4.16}$$

*with $\gamma$ defined in (4.13);*

3. $C_\alpha := \varepsilon(0)\max_{t\in[0,+\infty)}\left\{\frac{\alpha(t)}{\varepsilon(t)}\right\}<\infty$;
4. $\varepsilon(0)$ *is chosen so that*

$$\frac{4N_1N_2+N_2\sqrt{\varepsilon(0)}}{2C_0}<\varepsilon(0)\min\left\{\frac{C_0}{2C_\alpha},\frac{1}{||y-z_0||}\right\}. \tag{4.17}$$

*Then Cauchy problem (2.6) has a unique solution $z(t)$ for $t\in[0,\infty)$ and*

$$||z(t)-y||\leq\frac{2C_0}{4N_1N_2+N_2\sqrt{\varepsilon(0)}}\varepsilon(t). \tag{4.18}$$

**Remark 4.4** If $T(\xi)$ is injective then Condition 1 is satisfied automatically.

**Remark 4.5** From (4.15) one can see that $\alpha$ depends on $y$ and cannot be known a priori. However it follows from condition 1 of Theorem 4.3 that $\alpha(t)\to 0$ as $t\to+\infty$. Therefore in numerical applications of this scheme one should try different functions $\varepsilon(t)$ (and different points $z_0$) to satisfy condition 3. Then, since $C_0$ and $C_\alpha$ are invariant with respect to multiplication of $\varepsilon(t)$ by a positive constant, one can choose $\varepsilon(0)$ sufficiently large in order to satisfy condition 4. Such a choice can be done for an arbitrary $z_0$.

**Proof of Theorem 4.3** Since $x(t) \equiv y$ in our case, one gets $\beta(t) = \alpha(t)$. Let us choose $\mu(t) := \frac{\lambda}{\varepsilon(t)}$. Conditions of Theorem 2.4 can be written as follows:

$$\frac{N_1 N_2}{\varepsilon(t)} + \frac{N_2}{4\sqrt{\varepsilon(t)}} \leq \frac{\lambda}{2\varepsilon(t)} \left(\gamma - \frac{|\dot{\varepsilon}(t)|}{\varepsilon(t)}\right), \tag{4.19}$$

$$\alpha(t) \leq \frac{\varepsilon(t)}{2\lambda} \left(\gamma - \frac{|\dot{\varepsilon}(t)|}{\varepsilon(t)}\right), \quad \lambda||y - z_0|| < \varepsilon(0). \tag{4.20}$$

Inequality (4.19) is equivalent to the following one:

$$\lambda \geq \frac{4N_1 N_2 + N_2\sqrt{\varepsilon(t)}}{2\left(\gamma - \frac{|\dot{\varepsilon}(t)|}{\varepsilon(t)}\right)}. \tag{4.21}$$

Take

$$\lambda := \frac{4N_1 N_2 + N_2\sqrt{\varepsilon(0)}}{2C_0}. \tag{4.22}$$

Then (4.21) follows from (4.22), from the monotonicity of $\varepsilon(t)$ and from (4.16). Inequality (4.17) implies that

$$\frac{4N_1 N_2 + N_2\sqrt{\varepsilon(0)}}{2C_0} < \frac{C_0}{2\max_{t\in[0,+\infty)}\left\{\frac{\alpha(t)}{\varepsilon(t)}\right\}} \tag{4.23}$$

If $\lambda$ is defined by (4.22), then one has:

$$\max_{t\in[0,+\infty)}\left\{\frac{\alpha(t)}{\varepsilon(t)}\right\} < \frac{C_0}{2\lambda}.$$

From (4.16) it follows that the first inequality in (4.20) holds. Finally, one obtains from (4.17) that

$$\frac{4N_1 N_2 + N_2\sqrt{\varepsilon(0)}}{2C_0} < \frac{\varepsilon(0)}{||y - z_0||},$$

which implies the second inequality in (4.20). Since $x(t) \equiv y$ and $\mu(t) := \frac{\lambda}{\varepsilon(t)}$, by (2.37) and (4.22) one concludes that (4.18) holds. $\square$

Consider now a variant of Gauss-Newton continuous method with $\Phi$ defined as follows:

$$\Phi(h,t) := -T^{-1}_{\varepsilon(t)}(h)\{F'^*(h)F(h) + \varepsilon(t)(h - z_0)\}. \tag{4.24}$$

Here $F$ is not assumed compact. The following lemma is a consequence of Theorem 2.3 in [**2**] (see also Theorem 2.4 in [**9**]).

**Lemma 4.6** *Assume that:*

1. *problem (1.1) has a unique solution $y$;*
2. *$\Phi$ is defined by (4.24), $F$ is twice Fréchet differentiable and estimates (3.13) hold;*
3. *$z_0$ is chosen so that, for some $v \in H$, one has:*

$$y - z_0 = T^{\zeta}(y)v, \quad \frac{1}{2} \leq \zeta \leq 1; \tag{4.25}$$

4. $\varepsilon(t) > 0$,
$$\left[\frac{1}{2}\varepsilon^{\zeta-\frac{1}{2}}(t)N_1\zeta^\zeta(1-\zeta)^{1-\zeta} + \frac{N_1^{2(\zeta+1)}}{N_1^2+\varepsilon(t)}\right]||v|| < 1,$$
*where*
$$(1-\zeta)^{1-\zeta}|_{\zeta=1} := \lim_{\zeta\to 1-0}(1-\zeta)^{1-\zeta} = 1.$$

*Then for $x(t) \equiv y$ estimate (2.30) holds with*
$$\alpha(t) := \varepsilon^\zeta(t)\zeta^\zeta(1-\zeta)^{1-\zeta}||v||, \quad \sigma(t) := \frac{N_2}{4\sqrt{\varepsilon(t)}}, \tag{4.26}$$

$$\gamma(t) := 1 - \frac{1}{2}\varepsilon^{\zeta-\frac{1}{2}}(t)N_2\zeta^\zeta(1-\zeta)^{1-\zeta}||v|| - \frac{N_1^{2(\zeta+\frac{1}{2})}N_2||v||}{N_1^2+\varepsilon(t)} > 0. \tag{4.27}$$

**Remark 4.7** The convergence theorem in the case $\zeta = 1$ for continuous Gauss-Newton method is proved in [**2**, Theorem 2.3]. For the discrete Gauss-Newton method the case $0 \le \zeta < \frac{1}{2}$ is analyzed in [**9**] under some additional assumptions on the operator. For the noise-free case it is shown that the rate of convergence is $o(\varepsilon^\zeta)$.

**Proof of Lemma 4.6** From (4.8) and (4.25) one gets
$$(\Phi(h,t), h-y) = -(T^{-1}_{\varepsilon(t)}(h)[F'^*(h)F'(h) + \varepsilon(t)(h-y) + \varepsilon(t)(y-z_0)], h-y)$$
$$\le -||h-y||^2 + \frac{N_2}{4\sqrt{\varepsilon(t)}}||h-y||^3 - \varepsilon(t)(T^{-1}_{\varepsilon(t)}(h)T^\zeta(y)v, h-y).$$
Following [**9**] one estimates the inner product:
$$(T^{-1}_{\varepsilon(t)}(h)T^\zeta(y)v, h-y) = (T^{-1}_{\varepsilon(t)}(h)[T(y)-T(h)]T^{-1}_{\varepsilon(t)}(y)T^\zeta(y)v, h-y) + (T^{-1}_{\varepsilon(t)}(y)T^\zeta(y)v, h-y).$$
From the spectral theorem for selfadjoint linear operator $T(\eta)$, and for any $\eta \in H$, one gets:
$$||T^{-1}_{\varepsilon(t)}(\eta)T^\zeta(\eta)|| \le \max_{s\ge 0}\frac{s^\zeta}{s+\varepsilon(t)} = \frac{\zeta^\zeta(1-\zeta)^{1-\zeta}}{\varepsilon^{1-\zeta}(t)}. \tag{4.28}$$
Since
$$T(h) - T(y) = F'^*(h)[F'(h) - F'(y)] + [F'^*(h) - F'^*(y)]F'(y)$$
and $\frac{1}{2} \le \zeta \le 1$, from the polar decomposition one gets the estimate
$$||T^{\frac{1}{2}}(\eta)T^{-1}_{\varepsilon(t)}(\eta)T^\zeta(\eta)|| \le \max_{N_1^2\ge s\ge 0}\frac{s^{\zeta+\frac{1}{2}}}{s+\varepsilon(t)} \le \frac{N_1^{2(\zeta+\frac{1}{2})}}{N_1^2+\varepsilon(t)},$$
which implies
$$||T^{-1}_{\varepsilon(t)}(h)[T(y)-T(h)]T^{-1}_{\varepsilon(t)}(y)T^\zeta(y)|| \le ||T^{-1}_{\varepsilon(t)}(h)F'^*(h)(F'(h)-F'(y))T^{-1}_{\varepsilon(t)}(y)T^\zeta(y)||$$
$$+||T^{-1}_{\varepsilon(t)}(h)(F'^*(h)-F'^*(y))F'(y)T^{-1}_{\varepsilon(t)}(y)T^\zeta(y)||$$
$$\le \left[\frac{N_2}{2\sqrt{\varepsilon(t)}}\frac{\zeta^\zeta(1-\zeta)^{1-\zeta}}{\varepsilon^{1-\zeta}(t)} + \frac{N_2}{\varepsilon(t)}\frac{N_1^{2(\zeta+\frac{1}{2})}}{N_1^2+\varepsilon(t)}\right]||h-y||.$$

Therefore one obtains:

$$(\Phi(h,t), h-y) \leq \varepsilon^{\zeta}(t)\zeta^{\zeta}(1-\zeta)^{1-\zeta}||v||\,||h-y||$$

$$-\left\{1-\frac{1}{2}\varepsilon^{\zeta-\frac{1}{2}}(t)N_2\zeta^{\zeta}(1-\zeta)^{1-\zeta}||v|| - \frac{N_1^{2(\zeta+\frac{1}{2})}N_2||v||}{N_1^2+\varepsilon(t)}\right\}||h-y||^2 + \frac{N_2}{4\sqrt{\varepsilon(t)}}||h-y||^3. \tag{4.29}$$

□

**Remark 4.8** Note that assumption (4.25) is not algorithmically verifiable. However, practitioners may try different $z_0$ and choose the one for which the algorithm works better, that is convergence is more rapid and the algorithm is more stable.

Assumptions of the type (4.25) (sourcewise representation) became popular recently, because they allow one to establish some error estimates for the approximate solution. But one has to remember that the results based on such assumptions are of limited value because one has no algorithm for choosing $z_0$ for which (4.25) holds, and $y$ in (4.25) is unknown.

If $T = T^*$ is compact and the null space $N(T) = \{0\}$, then the range $R(T)$ is dense in $H$, so in any neighborhood of $y$ there are points $z_0$ for which (4.25) holds. On the other hand, since $R(T)$ is not closed in the same neighborhood there are also points $z_0$ for which (4.25) fails to hold. This is why the methods for solving nonlinear ill-posed problems, based on the assumption (4.25) or similar assumptions are not quite satisfactory although they might work in practice sometimes, for reasons which are yet not clear.

In general, in order to get a convergence theorem in an ill-posed case one needs some additional assumptions on the Fréchet derivative of the operator $F$, for example condition (4.6), or (4.25), or some other condition ( see e.g., [**13**], condition (2.11)).

**Theorem 4.9** *Let the assumptions of Lemma 4.6 be satisfied and*

1. $\varepsilon(t)$ *is continuously differentiable, monotonically tends to* $0$, *and*

$$C_0 := \min_{t\in[0,+\infty)}\left\{\gamma(t) - \zeta\frac{|\dot{\varepsilon}(t)|}{\varepsilon(t)}\right\} > 0, \tag{4.30}$$

*with* $\gamma(t)$ *defined in (4.27);*

2. $z_0$ *and* $\varepsilon(0)$ *are chosen so that*

$$\frac{N_2\varepsilon^{\zeta-\frac{1}{2}}(0)}{2C_0} < \min\left\{\frac{C_0}{2\zeta^{\zeta}(1-\zeta)^{1-\zeta}||v||}, \frac{\varepsilon^{\zeta}(0)}{||y-z_0||}\right\}. \tag{4.31}$$

*Then the Cauchy problem (2.6) has a unique solution* $z(t)$ *for* $t \in [0,+\infty)$ *and*

$$||z(t)-y|| \leq \frac{2C_0}{N_2\varepsilon^{\zeta-\frac{1}{2}}(0)}\varepsilon^{\zeta}(t). \tag{4.32}$$

**Proof** Choose $\mu(t) := \frac{\lambda}{\varepsilon^{\zeta}(t)}$. Conditions of Theorem 2.4 can be rewritten as follows:

$$\frac{N_2}{4\sqrt{\varepsilon(t)}} \leq \frac{\lambda}{2\varepsilon^{\zeta}(t)}\left(\gamma(t) - \zeta\frac{|\dot{\varepsilon}(t)|}{\varepsilon(t)}\right), \tag{4.33}$$

$$\zeta^{\zeta}(1-\zeta)^{1-\zeta}||v|| \le \frac{1}{2\lambda}\left(\gamma(t) - \zeta\frac{|\dot{\varepsilon}(t)|}{\varepsilon(t)}\right), \quad \frac{\lambda||y-z_0||}{\varepsilon^{\zeta}(0)} < 1. \tag{4.34}$$

Inequality (4.33) is equivalent to the following one:

$$\frac{N_2\varepsilon^{\zeta-\frac{1}{2}}(t)}{2\left(\gamma(t) - \zeta\frac{|\dot{\varepsilon}(t)|}{\varepsilon(t)}\right)} \le \lambda. \tag{4.35}$$

If one takes

$$\lambda := \frac{N_2\varepsilon^{\zeta-\frac{1}{2}}(0)}{2C_0}, \tag{4.36}$$

then (4.35) follows from (4.36), from the monotonicity of $\varepsilon(t)$ and from (4.30). Inequality (4.31) implies that

$$\frac{N_2\varepsilon^{\zeta-\frac{1}{2}}(0)}{2C_0} < \frac{C_0}{2\zeta^{\zeta}(1-\zeta)^{1-\zeta}||v||}. \tag{4.37}$$

For $\lambda$ defined by (4.36) inequality (4.37) can be written as

$$\zeta^{\zeta}(1-\zeta)^{1-\zeta}||v|| < \frac{C_0}{2\lambda}. \tag{4.38}$$

From (4.38) one obtains the first inequality (4.34). Finally, from (4.31) one concludes that

$$\frac{N_2\varepsilon^{\zeta-\frac{1}{2}}(0)}{2C_0} < \frac{\varepsilon^{\zeta}(0)}{||y-z_0||},$$

which is equivalent to the second inequality (4.34) for $\lambda$ defined by (4.36). Since $x(t) \equiv y$ and $\mu(t) := \frac{\lambda}{\varepsilon^{\zeta}(t)}$, by (2.37) and (4.36) one gets estimate (4.32). $\square$

**Remark 4.10** One can take $x(t)$ in Theorem 4.9 as the minimizer of the problem

$$||\,F(x)||^2 + \varepsilon(t)||x-z_0||^2 = \inf,\ x \in H, \quad 0 < \varepsilon(t) \to 0,\ \text{as } t \to +\infty, \tag{4.39}$$

instead of taking $x(t) \equiv y$. Problem (4.39) is solvable for $w$-continuous operator $F$ and $\lim_{t\to\infty}||x(t)-y|| = 0$. If one can obtain the estimate:

$$||F(x(t))|| = O(\varepsilon(t)), \tag{4.40}$$

then one can prove that

$$||\dot{x}(t)|| = O\left(\frac{|\dot{\varepsilon}(t)|}{\varepsilon(t)}\right) \tag{4.41}$$

and estimate (2.30) holds. However we do not have examples of nonlinear operators $F(x)$ satisfying estimate (4.40) and such that $F'(y)$ is not boundedly invertible.

## 5 Numerical Example

The aim of this section is to illustrate the efficiency of scheme (2.6) with $\Phi$ defined in (3.2) and in (4.24) for solving a practically interesting ill-posed nonlinear equation.

Consider the following iterative process

$$x^{k+1} = f_{\mu,z}(x^k)$$

for a differentiable function $f_{\mu,z}$ depending on two parameters $\mu$ and $z$ with the only maximum at the point $\hat{x}$. This process is characterized by the Feigenbaum constants:

$$\delta_z = \lim_{j\to\infty} \frac{\mu_{z_j} - \mu_{z_{j-1}}}{\mu_{z_{j+1}} - \mu_{z_j}}, \quad \alpha_z = \lim_{j\to\infty} \frac{d_{z_j}}{d_{z_{j+1}}},$$

where $\mu_j$ are the critical values of parameter $\mu$, for which a doubling of the period of the function $f_{\mu,z}$ occurs (the appearance of $2^j$ cycle), and $d_{z_j}$ is the algebraic distance (could be negative) between zero and the nearest attractor (the limit point in $2^j$ cycle). The calculation of $\alpha_z$ is a problem of a practical interest because it is not known yet if they satisfy any algebraic relations or not. As it is shown in [**15**], $\alpha_z$ can be found from the following nonlinear functional equation:

$$g_z(x) = -\alpha_z g_z\left(g_z\left(\frac{x}{\alpha_z}\right)\right) \tag{5.1}$$

with the unknown function $g_z$ and the initial condition $g(0) = 1$. Then

$$\alpha_z = -\frac{1}{g_z(1)}. \tag{5.2}$$

Functional equation (5.1) does not have in general a unique solution. In [**10**] numerical results are given which suggest, according to [**10**], that in certain restricted classes of analytic functions the solution to (5.1) is unique.

In [**10**] the constants $\alpha_z$ are computed with high accuracy on a class of even concave functions analytic on $[-1, 1]$ for integer $z$, $2 \le z \le 12$. Approximate solutions of (5.1) are constructed as polynomial approximations:

$$g_z(x) = 1 + \sum_{i=1}^{n} q_i |x|^{zi},$$

where $q_i$ are the solutions to the following nonlinear system:

$$F_j := \left(1 + \sum_{i=1}^{n} q_i\right)\left(1 + \sum_{i=1}^{n} q_i |x_j|^{zi}\right) - 1 - \sum_{i=1}^{n} q_i \left|1 + \sum_{i=1}^{n} q_i \left|(1 + \sum_{i=1}^{n} q_i)x_j\right|^{zi}\right|^{zi} = 0, \tag{5.3}$$

where $x_j$ are obtained from the partition of the segment $[0, 1]$.

In [**10**] classical Newton's method is successfully applied to a numerical solution of system (5.3) and computation of $\alpha_z$ for $z = 2, \ldots, 12$. For $z > 12$ constants $\alpha_z$ are not found in [**10**].

The goal of our experiment is to calculate $\alpha_z$ for $z = 2, \ldots, 12$ (and to compare with [**10**]) and also for $z > 12$ using schemes (2.6) – (3.2) and (2.6) – (4.24). The function $g_z(x)$

is even, therefore it is sufficient to find it on $[0,1]$. Since uniform partition of $[0,1]$ works for small $z$ only ($z = 2, 3$), the nonlinear partition $x_j := \left(\frac{j}{n}\right)^{\frac{1}{z}}$ is chosen.

The Jacobi matrix

$$F' := \left[\frac{\partial F_j}{\partial q_l}\right]_{j,l=\overline{1,n}} = |x_j|^{zl}\left(1+\sum_{i=1}^{n} q_i\right) + \left(1+\sum_{i=1}^{n} q_i |x_j|^{zi}\right) - \left|1+\sum_{i=1}^{n} q_i \left|(1+\sum_{i=1}^{n} q_i)x_j\right|^{zi}\right|^{zl}$$
$$-\sum_{i=1}^{n} q_i zi \left|1+\sum_{i=1}^{n} q_i \left|(1+\sum_{i=1}^{n} q_i)x_j\right|^{zi}\right|^{zi-1} \left\|\left(1+\sum_{i=1}^{n} q_i\right)x_j\right|^{zl} + \sum_{i=1}^{n} q_i zi \left|\left(1+\sum_{i=1}^{n} q_i\right)x_j\right|^{zi-1}\right|$$

is strictly ill-posed for $z \geq 2$ and $n \geq 2$. The condition number for any fixed $n$ increases about ten times as $z$ is replaced by $z+1$. Therefore solving (5.3) for large $z$ and $n \geq 2$ is a very unstable problem to which the standard numerical methods are not applicable. However our methods, based on Theorems 2.4, 3.10 and 4.9, do work and yield the Feigenbaum constants $\alpha_z$ for $z = 2, \ldots, 26$.

For a more accurate approximation of $g_z$ one has to take $n$ large enough, but then the problem of the choice of an initial approximation occurs: for $z = 2$, $n = 2$ or $n = 3$ system (5.3) has many solutions. By this reason the scheme described in [**7**] is used. First, system (5.3) is solved for $n = 1$, then the solution of (5.3) with $n = 1$ is taken as the initial guess for the case $n = 2$, etc. When $z = 2$, $n = 1$, $x = 1$ system (5.3) is reduced to one algebraic equation with respect to $q_1$:

$$q_1(q_1+1)(q_1^5 + 3q_1^4 + 3q_1^3 + 3q_1^2 + 2q_1 - 1) = 0$$

and the two obvious solutions are $q_1^{(1)} = 0$, $q_1^{(2)} = -1$. Since the function $g_z(x)$ is even and concave, the initial condition $g_z(0) = 1$ implies $g_z(1) < 1$, that is $1 + q_1 < 1$, $q_1 < 0$. Therefore one has to find the negative roots of the equation:

$$q_1^5 + 3q_1^4 + 3q_1^3 + 3q_1^2 + 2q_1 - 1 = 0.$$

Such roots are: $q_1^{(3)} = -1.8597174\ldots$, $q_1^{(4)} = -1.4021968\ldots$. Thus for the system of two equations ($z = 2$, $n = 2$) the initial data are:

1. $q_1 = -1, \quad q_2 = 0;$
2. $q_1 = -1.8597174\ldots, \quad q_2 = 0;$
3. $q_1 = -1.4021968\ldots, \quad q_2 = 0.$

In the first two cases the solutions to (5.3) ($z = 2$, $n = 2$) are not concave on $[-1,1]$. In the third case the graph of the polynomial is concave and

$$g_2(x) \approx 1 - 1.5416948x^2 + 0.1439197x^4.$$

For the system of three equations ($z = 2$, $n = 3$) the initial data are:

$$q_1 = -1.5416948, \quad q_2 = 0.1439197, \quad q_3 = 0.$$

Then we continue this process. The maximum dimension we take is $n = 12$. If $n = 13$, the discrepancy is not less than for $n = 12$, and after $n = 14$ it grows.

For $z = 3$ we begin the computation with one equation ($n = 1$) also. As the initial approximation $q_1 = -1.4021968$ is taken, that is the solution to (5.3) with $z = 2$, $n = 1$. The dimension increases step by step till the discrepancy improves. For $z = 3$, $n = 1$ the solution to (5.3) with $z = 3$, $n = 1$ is used, etc. In our experiment $\alpha_z$ for $z = 2, \dots, 26$ are found. For $z = 2, \dots, 12$ they coincide with $\alpha_z$ proposed in [**10**]. Below the values of $\alpha_{13} - \alpha_{26}$ are presented (the values of $\alpha_2 - \alpha_{12}$ can be found in [**10**]). As the exact digits the ones that were the same as the result of both regularized procedures were taken.

$$\alpha_{13} = -1.22902, \quad \alpha_{14} = -1.21391, \quad \alpha_{15} = -1.20072, \alpha_{16} = -1.18910,$$

$$\alpha_{17} = -1.17879, \quad \alpha_{18} = -1.16957, \quad \alpha_{19} = -1.1612, \alpha_{20} = -1.1537,$$

$$\alpha_{21} = -1.1469, \quad \alpha_{22} = -1.140, \quad \alpha_{23} = -1.134, \alpha_{24} = -1.129,$$

$$\alpha_{25} = -1.124, \quad \alpha_{26} = -1.12.$$

Our numerical results indeed demonstrate the efficiency of procedures (2.6) – (3.2) and (2.6) – (4.24) and give Feigenbaum's constants for much larger range than in [**10**], which is of some practical interest. Contrary to the original conjecture [**15**] our numerical results confirm the conclusion of [**10**], which says that the Feigenbaum constants in fact depend on the parameter $z$.

## 6 Appendix

Here we prove a lemma about nonlinear differential inequalities. As we have shown in the previous sections, such inequalities are very useful in applications.

**Lemma 6.1** *Let $u \in C^1[0, +\infty)$, $u(t) \geq 0$ for $t > 0$, and*

$$\dot{u} \leq -a(t) f(u(t)) + b(t), \quad \text{for} \quad t > 0, \quad u(0) = u_0. \tag{6.1}$$

*Assume:*

*1) $a(t), b(t) \in C[0, +\infty)$, $a(t) > 0$, $b(t) \geq 0$ for $t > 0$,*

*2) $\int^{+\infty} a(t) dt = +\infty$, $\frac{b(t)}{a(t)} \to 0$ as $t \to +\infty$,*

*3) $f \in C[0, +\infty)$, $f(0) = 0$, $f(u) > 0$ for $u > 0$,*

*4) there exists $c > 0$ such that $f(u) \geq c$ for $u \geq 1$.*

*Under these assumptions (6.1) implies*

$$u(t) \to 0 \quad \text{as} \quad t \to +\infty. \tag{6.2}$$

**Remark 6.2** This Lemma is essentially Lemma 1 from [**5**]. We have added condition 4) and changed the proof slightly. Condition 4) is omitted in [**5**]. Without condition 4 the conclusion of Lemma 1 in [**5**] is false as we show by a counterexample at the end of this Appendix. Condition 4) is equivalent to the condition $f(w) \to 0$ implies $w \to 0$.

Assumptions about smoothness of $a(t)$ and $b(t)$ in [**5**] are not formulated. In Lemma 6.1 we assume that these functions are continuous and $a(t) > 0$. The continuity assumption can be relaxed, but in applications it is not restrictive, since we deal with the inequality.

**Proof** By assumptions 1), 2), the new variable $s = s(t) := \int\limits_0^t a(\tau) d\tau$, maps $t \in [0, +\infty)$ onto $s \in [0, +\infty)$. Write (6.1) as

$$\frac{dw}{ds} \leq -f(w(s)) + \beta(s), \quad \text{for} \quad s > 0, \quad w(0) = u_0, \tag{6.3}$$

where $w(s) := u(t(s))$ and $\beta(s) = \frac{b(t(s))}{a(t(s))} \to 0$ as $s \to +\infty$.

The lemma is proved if one proves

$$w(s) \to 0 \quad \text{as} \quad s \to +\infty. \tag{6.4}$$

Let $\kappa(s) \in C[0, +\infty)$ be an arbitrary function such that $\kappa(s) \geq 0$, $\kappa(s) \to 0$ as $s \to +\infty$ and $\int^{+\infty} \kappa(s)ds = +\infty$. For example one can take $\kappa(s) = \frac{1}{s+1}$. Define subsets of $\mathbf{R}_+ := \{s : s \geq 0\}$ as follows:

$$E := \{s : s > 0, f(w(s)) - \beta(s) \leq \kappa(s)\}, \quad F := \mathbf{R}_+ \setminus E. \tag{6.5}$$

**Claim:**

$$\sup E = +\infty. \tag{6.6}$$

We prove (6.6) later.

Assuming (6.6), consider $s_1 \in E$, $(s_1, s_2) \subset F$. Then

$$-f(w(s)) + \beta(s) < -\kappa(s) \text{ for } s_1 < s < s_2. \tag{6.7}$$

From (6.3) and (6.7) one gets

$$\frac{dw}{ds} < -\kappa(s) \text{ for } s_1 < s < s_2. \tag{6.8}$$

Therefore for $s_1 < s < s_2$ one has

$$w(s) \leq w(s_1) - \int_{s_1}^{s} \kappa(\tau)d\tau \leq w(s_1) \text{ for } s_1 < s < s_2, \quad s \in F. \tag{6.9}$$

Since $s_1 \in E$ one has

$$f(w(s_1)) \leq \kappa(s_1) + \beta(s_1) \to 0 \text{ as } s_1 \to +\infty, \quad s_1 \in E. \tag{6.10}$$

Here we have used the assumptions $\kappa(s) \to 0$ and $\beta(s) \to 0$ as $s \to +\infty$.

From (6.10) and assumption 4) it follows, that

$$w(s_1) \to 0 \quad \text{as} \quad s_1 \to +\infty, \quad s_1 \in E, \tag{6.11}$$

and from (6.9) and (6.11) it follows that

$$w(s) \to 0 \quad \text{as} \quad s \to +\infty, \quad s \in F. \tag{6.12}$$

Thus, to prove Lemma it is sufficient to prove (6.6).

Suppose (6.6) is false, that is $\sup E = s_3 < +\infty$. Then

$$f(w(s)) - \beta(s) > \kappa(s) \quad \text{for} \quad s > s_3. \tag{6.13}$$

From (6.3) and (6.13) one gets

$$\frac{dw}{ds} \leq -\kappa(s) \quad \text{for} \quad s > s_3. \tag{6.14}$$

Thus

$$w(s) \leq w(s_3) - \int_{s_3}^{s} \kappa(\tau)d\tau \to -\infty \quad \text{as} \quad s \to +\infty, \tag{6.15}$$

where we have used the assumption $\int^{+\infty} \kappa(s)ds = +\infty$. This contradicts the assumption $u \geq 0$ and proves Lemma 6.1. $\square$

The following example (which is a counterexample to Lemma 1 in [**5**]) shows that condition 4) of Lemma 6.1 is essential.

Take

$$f(u) = \begin{cases} u, & \text{for} \quad 0 \leq u \leq 1, \\ e^{1-u}, & \text{for} \quad 1 \leq u < +\infty, \end{cases} \qquad a(t) \equiv 1, \quad b(t) = \frac{3}{t+c}, \tag{6.16}$$

where $c > e^{-1}$ is an arbitrary constant.

One can check immediately that

$$u(t) = 1 + log(t + c) \tag{6.17}$$

satisfies inequality (6.1). The choice $c > e^{-1}$ guarantees that $u(t) > 0$ for all $t \geq 0$. Clearly $u(t) \to +\infty$ as $t \to +\infty$, so that conclusion (6.2) of Lemma 6.1 is false if condition 4) is omitted.

## Acknowledgments

The authors thank Professor Ya. Alber for useful remarks and Professor V. Vasin for a discussion of the numerical example.

## References


[1] Airapetyan, R.G. *Continuous Newton method and its modification*, Applicable Analysis, (to appear).

[2] Airapetyan, R.G., Ramm A.G. and Smirnova, A.B. [1999] *Continuous analog of Gauss-Newton method*, Math. Models and Meth. in Appl. Sci., **9**, N3, 463–474.

[3] Alber, Ya.I. [1975] *On a solution of operator equations of the first kind with accretive operators in Banach spaces,* Diffferen. Uravneniya, **11**, N12, 2242–2248.

[4] Alber, Ya.I. [1993] *The regularization method for variational inequalities with nonsmooth unbounded operators in Banach space,* Appl. Math. Lett., **6**, N4, 63–68.

[5] Alber, Ya.I. [1994] *A new approach to the investigation of evolution differential equations in Banach spaces,* Nonlin. Anal., Theory, Methods & Appl., **23**, N9, 1115–1134.

[6] Argyros, I.K. [1998] *Polynomial operator equations in abstract spaces and applications,* CRC Press, Boca Raton.

[7] Babenko, K.I. [1986] *Fundamentals of the numerical analysis,* Nauka, Moscow.

[8] Beckenbach, E. and Bellman R. [1961] *Inequalities,* Springer-Verlag, Berlin.

[9] Blaschke, B., Neubauer, A. and Scherzer O. [1997] *On convergence rates for the iteratively regularized Gauss-Newton method,* IMA J. Num. Anal., **17**, 421–436.

[10] Briggs, K [1991] *A precise calculation of the Feigenbaum constants,* Mathematics of computations, **57**, N195, 435–439.

[11] Decker, D.W., Keller, H.B. and Kelley, C.T. [1983] *Convergence rates for Newton's method at singular points,* SIAM J. Numer. Anal., **20**, N2, 296–314.

[12] Deimling, K. [1985] *Nonlinear functional analysis,* Springer-Verlag, New York.

[13] Deuflhard, P., Engl, H.W. and Scherzer, O. [1998] *A convergence analysis of iterative methods for the solution of nonlinear ill-posed problems under affinely invariant conditions,* Inv. Probl., **14**, 1081–1106.

[14] Engl, H.W., Hanke, M. and Neubauer, A. [1996] *Regularization of inverse problems,* Kluwer Acad. Publ. Group, Dordrecht.

[15] Feigenbaum, M.J. [1978] *Quantitative of universality for a class of nonlinear transformations,* J. Stat. Phys. **19**, N1, 25–52. New York.

[16] M.K. Gavurin, [1958] *Nonlinear functional equations and continuous analogies of iterative methods,* Izv. Vuzov. Ser. Matematika. **5** (1958), 18–31.

[17] Kamke, E. [1974] *Differentialgleichungen. Lösungmethoden und Lösungen,* Chelsea, New York.

[18] Ryazantseva, I.P. [1994] *On some continuous regularization methods for monotone equations,* Comput. Math. Math. Phys., **34**, N1, 1–7.

[19] Szarski, J. [1967] *Differential inequalities,* PWN, Warszawa.

[20] Vasin, V.V. and Ageev, A.L., [1995] *Ill-posed problems with a priori information*, VNU, Utrecht.