SINGULARITIES OF THE RADON TRANSFORM

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The singular support of the Radon transform Rf of a function f with a compact support is described in terms of the Generalized Legendre transform. The asymptotic behavior of Rf near the singular support modulo smooth functions is described in terms of a geometrical invariant, introduced in the paper.

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1 Introduction

Let N be a positive number, $D \subset \mathbf{R}^n$ be a compact domain bounded by a piecewise-C^N surface S, and $f \in L^1(\mathbf{R}^n)$ be a function such that $\text{supp}(f) = D$, where $\text{supp}(f)$ is the support of f, $f|_D \in C^N(D)$ and $f|_S \neq 0$. One says that S is piecewise-C^N if S is a union

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of a finite number of submanifolds $S_{\{1,\dots,m_j\}}$ of codimensions $m_j \geq 1$ and every $S_{\{1,\dots,m_j\}}$ is locally parametrized by C^N functions.

Consider the Radon transform $\hat{f}$ of f:

$$\hat{f}(\alpha,p) := Rf(\alpha,p) := \int\limits_{p-\alpha\cdot x=0} f(x)\mu(x), \quad \hat{f}(\lambda\alpha,\lambda p) = \hat{f}(\alpha,p), \quad \lambda \neq 0, \quad \lambda \in \mathbf{R}, \tag{1.1}$$

where $\mu(x)$ is the Lebesgue measure on the hyperplane $\alpha \cdot x - p = 0$, induced by the Lebesgue measure in $\mathbf{R}^n$, $p \in \mathbf{R}$, $\alpha \in \mathbf{R}^n \setminus \{0\}$. The Radon transform can be written also in the following equivalent form:

$$Rf(\alpha,p) = |\alpha| \int\limits_{\mathbf{R}^n} f(x)\delta(\alpha\cdot x - p)dx. \tag{1.2}$$

The goal of this paper is to describe the singular support $\hat{S}$ of $\hat{f}(\alpha,p)$ and the asymptotic behavior of $\hat{f}(\alpha,p)$ near its singular support.

The Radon transform is a Fourier Integral Operator (FIO) which maps $\mathcal{E}'(\mathbf{R}^n)$ into $D'(\mathbf{P}^n)$, where $\mathbf{P}^n$ is the projective space. The wave front set $WF(\hat{f})$ of $\hat{f}(\alpha,p)$ is the image of $WF(f)$ under the mapping V which acts from $\dot{\mathbf{T}}^*\mathbf{R}^n$ to $\dot{\mathbf{T}}^*\mathbf{P}^n$, where $\dot{\mathbf{T}}^*\mathbf{R}^n$ and $\dot{\mathbf{T}}^*\mathbf{P}^n$ are the cotangent bundles (without zero section) over $\mathbf{R}^n$ and $\mathbf{P}^n$ correspondingly. The mapping V is defined in Sect. 2, formulas (2.7)–(2.8), (see [7, p. 461] for references). This mapping V induces the corresponding multivalued mapping $V_* : \mathbf{R}^n \to \mathbf{P}^n$:

$$\begin{array}{ccc} \dot{\mathbf{T}}^*\mathbf{R}^n & \xrightarrow{V} & \dot{\mathbf{T}}^*\mathbf{P}^n \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ \mathbf{R}^n & \xrightarrow{V_*} & \mathbf{P}^n \end{array}$$

defined as:

$$V_* = \pi_2 \circ V \circ \pi_1^{-1}, \tag{1.3}$$

The map V_* sends singsupp(f) *onto* singsupp$(\hat{f})$ (Theorem 2.1).

Because $f|_S \neq 0$ one has $S \subset$ singsupp(f). We study the singularities generated by the boundary S assuming that $f \in C^N(D)$ for some positive N. The interest to such singularities comes from local tomography [5], where one has to determine the discontinuities of a piecewise-smooth function from those of its Radon transform. In order to describe the singularities generated by the boundary we introduce a mapping $\mathcal{L}$ which sends submanifolds of $\mathbf{R}^n$ onto varieties in $\mathbf{P}^n$.

Let $S_{\{1,\ldots,m\}}$ denote an intersection of C^1-manifolds $S_1, S_2, \ldots, S_m$, each of which is a hypersurface (codimension-one C^1-manifold). For a given C^1 submanifold $S_{\{1,\ldots,m\}}$ in S of codimension $m \geq 1$ denote by $N^*S_{\{1,\ldots,m\}}$ the conormal bundle on S in $\dot{\mathbf{T}}^*\mathbf{R}^n$. Then V induces the multivalued mapping $\mathcal{L} : S_{\{1,\ldots,m\}} \rightarrow \mathbf{P}^n$:

$$\begin{array}{ccc} N^*S_{\{1,\ldots,m\}} & \xrightarrow{V} & \dot{\mathbf{T}}^*\mathbf{P}^n \\ \pi_S \downarrow & & \downarrow \pi_2 \\ S_{\{1,\ldots,m\}} & \xrightarrow{\mathcal{L}} & \mathbf{P}^n \end{array}$$

defined as:

$$\mathcal{L} = \pi_2 \circ V \circ \pi_S^{-1}, \tag{1.4}$$

where π_S is the natural projection of $N^*S_{\{1,\ldots,m\}}$. Because S is a union of $S_{\{1,\ldots,m_j\}}$ the mapping $\mathcal{L}$ is defined on S.

If $f_D \in C^\infty(D)$ then singsupp$(f) = S$ and *the map* $\mathcal{L}$ *sends* singsupp(f) *onto* singsupp$(\hat{f})$ (Theorem 3.4).

The mapping $\mathcal{L}$ has a simple geometrical meaning: it sends a point $\overline{y} \in S$ onto the the tangent plane to S passing through the point $\overline{y}$ if S is C^1 at the point $\overline{y}$ (in this case $\overline{y}$

belongs to a submanifold of codimension one), and onto a bundle of planes otherwise, thus $\mathcal{L}$ is multivalued in the latter case. If $\overline{y} \in S$ is a point on a C^1 submanifold of codimension one then the tangent plane to S at the point $\overline{y}$ is defined as usual: $x_n - \overline{y}_n = \nabla g(\overline{y}') \cdot (x' - \overline{y}')$, where $x_n = g(x')$, $x' := (x_1, \dots, x_{n-1})$, is the local equation of S in a neighborhood of the point $\overline{y}$. If $\overline{y} \in S$ is a point on a C^1 submanifold $S_{\{1,\dots,m\}}$ in S of codimension $m > 1$, then the tangent plane to S at $\overline{y}$ is, *by definition*, any plane passing through $\overline{y}$ and containing the tangent space to $S_{\{1,\dots,m\}}$. For example, two surfaces S_1 and S_2 in $\mathbf{R}^3$ may intersect over a curve $S_{\{1,2\}}$, and the tangent space to $S_{\{1,2\}}$ at a point $\overline{y}$ is a bundle of planes passing through the straight line tangent to $S_{\{1,2\}}$ at the point $\overline{y}$.

It turns out that $\mathcal{L}$ is a generalization of the Legendre transform for functions not necessarily smooth. Indeed it is shown in Sect. 3 that, if S is a smooth hypersurface in a neighborhood of the point $\overline{y}$, which is locally represented by the equation $x_n = g(x_1, \dots, x_{n-1})$, then its image under the Generalized Legendre transform (GLT), $\mathcal{L}(S)$, in inhomogeneous coordinates is given, in a neighborhood of the point $\mathcal{L}(\overline{y})$, by the equation $p = h(\alpha_1, \dots, \alpha_{n-1}, 1)$, where h is an image of g under the usual Legendre transform. One says that $h(\beta)$ is the Legendre transform of a C^1 function $g(x)$ if $h(\beta) = \beta \cdot x(\beta) - g(x(\beta))$, where $x(\beta)$ is the unique solution to the equation $\nabla g(x) = \beta$ (the classical Legendre transform is defined under the assumption that this equation is uniquely solvable). The inhomogeneous coordinates are defined by the formulas $\beta = -\alpha'/\alpha_n$, $\alpha' = (\alpha_1, \dots, \alpha_{n-1})$, $q = -p/\alpha_n$ if $\alpha_n \neq 0$.

The Generalized Legendre transform introduced in [5], [6] as a transformation acting on functions given on a smooth submanifold in $\mathbf{R}^n$ (not necessarily hypersurface) is a particular case of our GLT. Thus we call $\mathcal{L}$ also the Generalized Legendre transform (GLT). One can also define GLT of a submanifold of $\mathbf{P}^n$ using inhomogeneuous coordinates. Then *the duality law*:

$$S = \mathcal{L}(\mathcal{L}(S)), \tag{1.5}$$

formulated in [5, p. 103] is still true, for example, in the case of a smooth strictly convex boundary (submanifold of codimension one). In general one has the following relation:

$$S \subset \mathcal{L}(\mathcal{L}(S)). \tag{1.6}$$

In the second part of the paper (Sect. 5) the asymptotic behavior of $\hat{f}(\alpha,p)$ near the point $(\overline{\alpha},\overline{p}) \in \mathcal{L}(S)$ is described modulo of C^{N_1} functions, where $N_1 = [(N+n+\bar{m})/2]$, $[a]$ is the integer part of a real number a, $\bar{m} = \min\{m_j\}$, m_j are codimensions of $S_{\{1,\dots,m_j\}}$ corresponding to the points $\overline{y}_j \in \mathcal{L}^{-1}(\overline{\alpha},\overline{p})$. We show that modulo C^{N_1} functions, $\hat{f}(\alpha,p)$ near the singular support is equivalent to linear combinations of the three functions p_+^ν, p_-^ν and $p^\nu \log|p|$, $\nu = \frac{n+m_j}{2} - 1$, multiplied by polynomials with respect to p-variable with coefficients depending on $\overline{\alpha}$ and $\overline{p}$. The analysis of singularities is done in an appropriate local coordinate system. In order to formulate the corresponding result in an invariant form we introduce a geometrical invariant which characterizes an oriented intersection of m hypersurfaces at a point on S (Sect. 4).

2 The Radon transform

One can write the Radon transform (1.2) as an elliptic FIO:

$$\hat{f}(\alpha,p) = Rf = \frac{|\alpha|}{2\pi} \int\limits_{\mathbf{R}^n} \int\limits_{\mathbf{R}} e^{i\phi(p,\alpha,y,\theta)} f(y)\,d\theta\,dy, \tag{2.1}$$

where

$$\phi(p,\alpha,y,\theta)=\theta(p-\alpha\cdot y), \tag{2.2}$$

(p,α) are the homogeneous coordinates in $\mathbf{P}^n$,

$$\hat{f}(\lambda\alpha,\lambda p)=\hat{f}(\alpha,p),\quad \phi(\lambda p,\lambda\alpha,y,\theta)=\lambda\phi(p,\alpha,y,\theta). \tag{2.3}$$

In [4] it was noted that the Radon transform can be considered as a FIO.

Denote by $\dot{\mathbf{T}}^*\mathbf{R}^n$ and $\dot{\mathbf{T}}^*\mathbf{P}^n$ the cotangent bundles without zero section over $\mathbf{R}^n$ and $\mathbf{P}^n$ correspondingly, and by (p,α,ξ_0,ξ) the homogeneous coordinates in $\dot{\mathbf{T}}^*\mathbf{P}^n$, where $(p,\alpha)\in\mathbf{R}^{n+1}\setminus\{0\}$, $\xi_0\in\mathbf{R}$, $\xi_0\neq 0$, $\xi\in\mathbf{R}^n$. Since points $(\lambda p,\lambda\alpha)$ are identified with (p,α) if $\lambda\neq 0$ one has:

$$T_{(p,\alpha)}\mathbf{P}^n=T_{(p,\alpha)}(\mathbf{R}^{n+1}\setminus\{0\})/L_{(p,\alpha)}, \tag{2.4}$$

where $L_{(p,\alpha)}$ is the one-dimensional subspace of $T_{(p,\alpha)}(\mathbf{R}^{n+1}\setminus\{0\})$ generated by the vector $p\partial/\partial p+\alpha\cdot\partial/\partial\alpha$. Thus

$$T^*_{(p,\alpha)}\mathbf{P}^n=\mathrm{annul}(L_{(p,\alpha)})\subset T^*_{(p,\alpha)}(\mathbf{R}^{n+1}\setminus\{0\}), \tag{2.5}$$

where $\mathrm{annul}(L_{(p,\alpha)})$ is the set of covectors $\xi_0dp+\xi\cdot d\alpha\in T^*_{(p,\alpha)}(\mathbf{R}^{n+1}\setminus\{0\})$ which vanish on the vectors in $L_{(p,\alpha)}$, that is,

$$p\xi_0+\alpha\cdot\xi=0,\quad \forall(\xi_0,\xi)\in\mathbf{R}^{n+1}. \tag{2.6}$$

Let us define the mapping

$$V:\dot{\mathbf{T}}^*\mathbf{R}^n\ni(y,\eta)\mapsto\{(p,\alpha,\xi_0,\xi)\}\in\dot{\mathbf{T}}^*\mathbf{P}^n, \tag{2.7}$$

where $\{(p,\alpha,\xi_0,\xi)\}$ is the equivalence class containing the element (p,α,ξ_0,ξ) defined by the following formulas:

$$p = y\cdot\eta, \quad \alpha = \eta, \quad \xi_0 = 1, \quad \xi = -y. \tag{2.8}$$

Then

$$V^{-1} : \dot{\mathbf{T}}^*\mathbf{P}^n \ni (p,\alpha,\xi_0,\xi) \mapsto (y,\eta) \in \dot{\mathbf{T}}^*\mathbf{R}^n, \tag{2.9}$$

is defined by the formulas:

$$y = -\xi, \quad \eta = \alpha. \tag{2.10}$$

Indeed, it follows from (2.6) that if $\{(p,\alpha,1,\xi)\} \in \dot{\mathbf{T}}^*\mathbf{P}^n$, then $(p,\alpha,1,\xi)$ is the image of the point (y,η) defined by (2.10).

Remark 1 *One can consider $\dot{\mathbf{T}}^*\mathbf{R}^n$ and $\dot{\mathbf{T}}^*\mathbf{P}^n$ as symplectic manifolds with the symplectic form $\sum d\eta_j \wedge dy_j$ on $\dot{\mathbf{T}}^*\mathbf{R}^n$ and the push forward of $d\xi_0 \wedge dp + \sum d\xi_j \wedge d\alpha_j$ under the natural projection $\mathbf{R}^{n+1} \to \mathbf{P}^n$ as the symplectic form on $\dot{\mathbf{T}}^*\mathbf{P}^n$. The push forward of $d\xi_0 \wedge dp + \sum d\xi_j \wedge d\alpha_j$ is well defined because of (2.6). Then V is a diffeomorphism preserving the symplectic structures defined by this forms (symplectomorphism). In the global theory of FIO the graph of V is called a Local Canonical Graph (see [7, p. 461]). Then the Radon transform R is a FIO associated with a Local Canonical Graph of V.*

Theorem 2.1 *One has*

$$WF(\hat{f}) = V(WF(f)). \tag{2.11}$$

Proof. First we show that $WF(\hat{f}) \subset V(WF(f))$. The phase (2.2) is nondegenerate, that is, for $\theta \neq 0$, $d_{\theta,p,\alpha}\phi \neq 0$ and $d_{\theta,y}\phi \neq 0$. Indeed,

$$d_{\theta,p,\alpha}\phi = (p - \alpha \cdot y, \theta, -\theta y) \neq 0 \text{ for } \theta \neq 0,$$

and

$$d_{\theta,y}\phi = (p - \alpha \cdot y, -\theta\alpha) \neq 0 \text{ for } \theta \neq 0.$$

Denote by Ω_ϕ the set $\{(p, \alpha, \xi_0, \xi)\}$ of the solutions of the following system of algebraic equations:

$$\begin{cases} \frac{\partial\phi}{\partial\theta} = 0 \\ \xi_0 = \frac{\partial\phi}{\partial p}, \quad \xi = \frac{\partial\phi}{\partial\alpha}, \\ \eta = -\frac{\partial\phi}{\partial y}, \quad (y, \eta) \in WF(f). \end{cases} \tag{2.12}$$

Thus, from (2.2) and from (2.12) one gets

$$p = \alpha \cdot y, \quad \xi_0 = \theta, \quad \xi = -\theta y, \quad \eta = \theta\alpha; \quad \Omega_\phi = \{\theta^{-1}\eta \cdot y, \theta^{-1}\eta, \theta, -\theta y\}|_{\forall\theta\neq 0}. \tag{2.13}$$

Since $\xi_0 = \theta \neq 0$ one can consider the intersection $\dot{\mathbf{T}}^*\mathbf{P}^n \cap \{\xi_0 = 1\}$ as a coordinate system in $\dot{\mathbf{T}}^*\mathbf{P}^n$. Therefore (see (2.7) and (2.8)) one has:

$$WF(f) \ni (y, \eta) \mapsto (y \cdot \eta, \eta, 1, -y) = V(y, \eta) \in \Omega_\phi. \tag{2.14}$$

From the theory of FIO (see [7, p. 334]) it follows that

$$WF(\hat{f}) \subset \Omega_\phi = V(WF(f)).$$

In order to show that $V(WF(f)) \subset WF(\hat{f})$ one can construct a parametrix R' of R as a FIO associated with a Local Canonical Graph V^{-1} (see [7, p. 461]). Such parametrix exists because R is an elliptic FIO with the amplitude function equal to one ([7, p. 467]). Thus one gets that

$$WF(f) = WF(R'\hat{f}) \subset V^{-1}(WF(\hat{f})). \tag{2.15}$$

To finish the proof, one applies V to both sides of (2.15). $\square$

3 The Generalized Legendre transform

Let us consider a piecewise-C^1 surface S, which is a disjoint union of submanifolds $S_{\{1,\dots,m_j\}}$ of codimensions $m_j \geq 1$. In this section it is not assumed that S is a boundary of some domain.

For every point $y \in S$ denote by $S_{\{1,\dots,m_j\}}(y)$ the submanifold containing y, by $N^*S_{\{1,\dots,m_j\}}(y)$ the conormal bundle on $S_{\{1,\dots,m_j\}}(y)$ in $\dot{\mathbf{T}}^*\mathbf{R}^n$ and by $\pi_S(y) : N^*S_{\{1,\dots,m_j\}}(y) \to S_{\{1,\dots,m_j\}}(y)$ the natural projection.

Definition 3.1 *The Generalized Legendre transform (GLT) is a multivalued mapping $\mathcal{L} := S \to \mathbf{P}^n$ defined by the following formula: for every $y \in S$ one has*

$$\mathcal{L}(y) = \{(p,\alpha) : (p,\alpha) \in \pi_2 \circ V \circ \pi_S^{-1}(y) \subset \mathbf{P}^n\}. \tag{3.1}$$

It follows imediately from this definition that GLT is local with respect to y, that is, the image of a point (y,η) depends only on the geometry of S in a small neighborhood of y.

Let us assume that a point $y_0 \in S$ belongs to a submanifold $S_{\{1,\dots,m\}} \subset S$ of codimension m and so $S_{\{1,\dots,m\}}$ can be represented by means of the equations $G_1(y) = 0, \dots, G_m(y) = 0$

in some neighborhood U_0 of a point y_0, where $G_j(y)$, $j = 1, \dots, m$, are twice continuously differentiable functions and the Hessian $H(\overline{y}) = ||\partial G_i(\overline{y})/\partial y_j||_{i=1,\dots,m;\, j=1,\dots,n}$ has rank m. Then

$$N^*(S_{\{1,\dots,m\}} \cap U_0) = \{(y, \eta) : y \in U_0, \eta \in \operatorname{span}(\nabla G_1(y), \dots, \nabla G_m(y)))\}. \tag{3.2}$$

Correspondingly

$$V(N^*(S_{\{1,\dots,m\}} \cap U_0)) =$$

$$= \{(p, \alpha, 1, -y) : y \in U_0, p = y \cdot \alpha, \alpha \in \operatorname{span}(\nabla G_1(y), \dots, \nabla G_m(y))\} \tag{3.3}$$

and

$$\mathcal{L}(S \cap U_0) = \{(p, \alpha) : p = y \cdot \alpha, \alpha \in \operatorname{span}(\nabla G_1(y), \dots, \nabla G_m(y)), y \in U_0\}. \tag{3.4}$$

$S_{\{1,\dots,m\}}$ can be considered as the intersection of m surfaces $S_1, \dots, S_m$. Denote an exterior normal vector to the surface S_j by $\mathbf{N}_j(y)$. Since (p, α) are the homogeneous coordinates one can replace $\nabla G_j(y)$ by $\mathbf{N}_j(y)$ and get the equivalent representation of the set (3.4):

$$\mathcal{L}(S \cap U_0) = \{(p, \alpha) : p = y \cdot \alpha, \alpha \in \operatorname{span}(\mathbf{N}_1(y), \dots, \mathbf{N}_m(y)), y \in U_0\}. \tag{3.5}$$

The set (3.5) has a simple geometrical meaning (see Figure 1): the image of a point $y \in S \cap U_0$ is a bundle of planes containing $T_y S_{\{1,\dots,m\}}$.

According to Definition 3.1 GLT is a mapping, which sends sufaces in $\mathbf{R}^n$ onto some varieties in $\mathbf{P}^n$. At the same time one can consider this mapping as a mapping defined on piecewise-C^1 functions: if the image under GLT of a graph of such a function g can be represented as a graph of some function h, then one calls h the image of g under GLT: $h = \mathcal{L}(g)$. In order to clarify such an approach consider $\mathcal{L}$ in some particular cases.

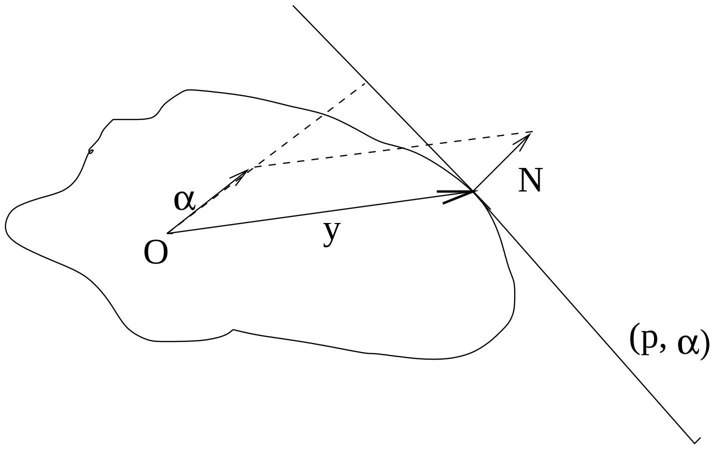

Figure 1: Plane (p,α) is a Legendre image of a point $y \in S_{\{1\}}$.

1. Let g be a C^1 function $y_n = g(y')$, $y' = (y_1, \ldots, y_{n-1})$, then $S = graph(g) := \{(y', y_n) : y_n = g(y')\}$ and from (3.4) one gets:

$$\mathcal{L}(S) = \{(p,\alpha) : p = y \cdot \alpha,\ \alpha = \lambda \nabla (g(y') - y_n),\ \lambda \in \mathbf{R}\}. \tag{3.6}$$

Denoting $\alpha' = (\alpha_1, \ldots, \alpha_{n-1})$ and choosing inhomogeneous coordinates $\beta = -\alpha'/\alpha_n$, $q = -p/\alpha_n$, $\alpha_n \neq 0$, one gets from (3.6) the following representation of $\mathcal{L}(S)$:

$$\mathcal{L}(S) = \{(q,\beta) : q = y' \cdot \beta - g(y'),\ \beta = \nabla_{y'} g(y')\}. \tag{3.7}$$

Assume that the equation $\beta = \nabla_{y'} g(y')$ is uniquely solvable for y' in a neighborhood $U_{\overline{y}'}$ of some point $\overline{y}'$, denote its solution by $y'(\beta)$, and define the function $q = h(\beta) := y'(\beta) \cdot \beta - g(y'(\beta))$ in $U_{\overline{y}'}$. Thus $\mathcal{L}(S)$ locally is the graph of the function $q = h(\beta)$, which is the image of the function $g(y')$ under the usual Legendre transform.

Thus if one considers the Legendre transform of smooth functions as the transform of the corresponding graphs (surfaces) then the usual Legendre transform is a localization of

GLT.

2. Consider a function $y_n = g(y')$ given on a submanifold $Y \subset \mathbf{R}^{n-1}$ of codimension $m-1$. Consider GLT $\mathcal{L}$ in a neighborhood of a point $\overline{y}' \in Y$ and assume that Y is locally parametrized by a mapping $y' = y'(u)$, $u \in \mathbf{R}^{n-m}$. Then $S = graph(g) := \{(y', y_n) : y_n = g(y'), y' \in Y\}$ is a submanifold of codimension m in $\mathbf{R}^n$ and

$$\mathcal{L}(S) = \{(p,\alpha) : p = y \cdot \alpha,\ \alpha = \lambda \nabla (g(y') - y_n) + \tilde{\alpha},\ \lambda \in \mathbf{R},\ \tilde{\alpha} \text{ is normal to } Y\}. \tag{3.8}$$

It follows from (3.8) that $\lambda = -\alpha_n$ and in a neighborhood $U_{\overline{y}}$ of the point $(\overline{y}', g(\overline{y}'))$ one has:

$$\sum_{k=1}^{n-1} \tilde{\alpha}_k \frac{\partial y'_k(u)}{\partial u_j} = 0 \text{ for } j = 1, \dots, n-1-m.$$

Thus for $(p,\alpha) \in S \cap U_{\overline{y}}$ one gets:

$$\sum_{k=1}^{n-1} \left(\alpha_k + \alpha_n \frac{\partial g(y')}{\partial y_k} \right) \frac{\partial y'_k(u)}{\partial u_j} = 0 \text{ for } j = 1, \dots, n-m,$$

and

$$\mathcal{L}(S \cap U_{\overline{y}}) = \{(p,\alpha) : p = y \cdot \alpha = \alpha_n g(y') + \alpha' \cdot y',\ \nabla_u(\alpha' \cdot y'(u) + \alpha_n g(y'(u))) = 0\}. \tag{3.9}$$

In the coordinates: $\beta = -\alpha'/\alpha_n$, $q = -p/\alpha_n$, one rewrites (3.9) as:

$$\mathcal{L}(S \cap U_{\overline{y}}) = \{(q,\beta) : q = y' \cdot \beta - g(y'),\ \nabla_u(\beta \cdot y'(u) - g(y'(u))) = 0\}. \tag{3.10}$$

Assume that the equation $\nabla_u(\beta \cdot y'(u) - g(y'(u))) = 0$ is uniquely solvable for y', denote its solution by $y'(\beta)$, and define the function $q = h(\beta) := y'(u(\beta)) \cdot \beta - g(y'(u(\beta)))$. Thus $\mathcal{L}(S)$ is the graph of the function $q = h(\beta)$, which is the image of $y_n = g(y')$ under the Generalized Legendre transform, defined in [6] (see also [5, p. 103]).

3. Let $g = (g_1, \ldots, g_m)$ be a C^1 vector function $z = g(x)$, $z \in \mathbf{R}^m$, $x \in \mathbf{R}^{n-m}$ and

$$S = graph(g) := \{y = (x, z) \,:\, x \in \mathbf{R}^{n-m},\, z \in \mathbf{R}^m,\, z = g(x)\}.$$

Then S is a C^1 submanifold of codimensions m in $\mathbf{R}^n$ and from (3.4) one gets:

$$\mathcal{L}(S) = \{(p, \alpha) : p = y \cdot \alpha, \alpha = \sum_{i=1}^{m} \lambda_i \nabla_{z,x}[g_i(x) - z_i]\}. \tag{3.11}$$

Therefore for $(p, \alpha) \in \mathcal{L}(S)$ one obtains

$$\alpha_{n-m+k} = -\lambda_i, \text{ and } \alpha_k = -\sum_{i=1}^{m} \alpha_{n-m+i} \frac{\partial g_i}{\partial y_k}, \quad k = 1, \ldots, m.$$

Denote $\alpha' = \{\alpha_1, \ldots, \alpha_m\}$, then

$$\mathcal{L}(S) = \{(p, \alpha) : \nabla_x[\alpha' \cdot x + \sum_{i=1}^{m} \alpha_{n-m+i} g_i(x)] = 0,\, p = \sum_{i=1}^{m} \alpha_{n-m+i} g_i(x) - \alpha' \cdot x\}.$$

Assume that the equation

$$\nabla_x \left[\alpha' \cdot x + \sum_{i=1}^{m} \alpha_{n-m+i} g_i(x)\right] = 0 \tag{3.12}$$

is uniquely solvable with respect to x in a small neighborhood of a point $\overline{x}$ and defines a function $x = x(\alpha)$.

Definition 3.2 *The Generalized Legendre transform of a C^1 vector-function g is the following function $p = h(\alpha)$:*

$$h(\alpha) := \mathcal{L}(g)(\alpha) := \sum_{i=1}^{m} \alpha_{n-m+i} g_i(x(\alpha)) - \alpha' \cdot x(\alpha). \tag{3.13}$$

4. Let us consider now a general case: g is a piece-wise C^1 vector function $z = g(x)$, $z \in \mathbf{R}^m$, $x \in \mathbf{R}^{n-m}$ and the graph of g is a stratified surface that is $graph(g)$ is a disjoint union of manifolds of different dimensions:

$$S = graph(g) := \{y = (x, z) \,:\, x \in \mathbf{R}^{n-m},\, z \in \mathbf{R}^m,\, z = g(x)\} = \cup S_{\{1,\ldots,m_j\}},$$

where $S_{\{1,\ldots,m_j\}}$ are C^1 submanifolds of codimensions m_j in $\mathbf{R}^n$. Consider a point $\overline{y} = (\overline{z}, \overline{x}) \in S$, then there exists a submanifold $S_{\{1,\ldots,m_j\}}$, $m_j \geq m$, containing $\overline{y}$. This submanifold can be represented as the graph of g restricted to a submanifold $X \subset \mathbf{R}^{n-m}$ of codimension $m_j - m$. If g is a C^1 vector function in a neighborhood of a point $\overline{x}$, then the corresponding $m_j = m$, so one has the situation of the Case 3. Assume that X is locally parametrized by a mapping $x = x(u)$, $u \in \mathbf{R}^{n-m_j}$. Then

$$\mathcal{L}(S_{\{1,\ldots,m_j\}}) = \{(p,\alpha) : p = y\cdot\alpha,\ \alpha = \sum_{k=1}^{m} \lambda_k \nabla(g_k(x) - z_k) + \tilde{\alpha},\ \tilde{\alpha} \text{ is normal to } X\}. \quad (3.14)$$

It follows from (3.14) that $\lambda_i = -\alpha_{n-m+1}$ for $i = 1, \ldots, m$. Thus in a neighborhood $U_{\overline{x},\overline{z}}$ of the point $(\overline{x}, \overline{z})$ one has:

$$\sum_{k=1}^{n-m} \tilde{\alpha}_k \frac{\partial x_k(u)}{\partial u_j} = 0 \text{ for } j = 1, \ldots, n - m_j.$$

Hence for $(p, \alpha) \in S_{\{1,\ldots,m_j\}} \cap U_{\overline{y}}$ one gets:

$$\sum_{k=1}^{n-m} \left(\alpha_k + \sum_{i=1}^{m} \alpha_{n-m+i} \frac{\partial g_i(x)}{\partial x_k}\right) \frac{\partial x_k(u)}{\partial u_j} = 0 \text{ for } j = 1, \ldots, n - m_j,$$

and

$$\mathcal{L}(S_{\{1,\ldots,m_j\}} \cap U_{\overline{y}}) = \{(p,\alpha) : p = y\cdot\alpha$$
$$= \alpha'\cdot x + \sum_{i=1}^{m} \alpha_{n-m+i} g_i(x),\ \nabla_u \left(\alpha'\cdot x(u) + \sum_{i=1}^{m} \alpha_{n-m+i} g_i(x(u))\right) = 0\right\}. \quad (3.15)$$

Assume that the equation

$$\nabla_u \left(\alpha'\cdot x(u) + \sum_{i=1}^{m} \alpha_{n-m+i} g_i(x(u))\right) = 0 \quad (3.16)$$

is uniquely solvable for u and defines a function $u = u(\alpha)$.

Definition 3.3 *The Generalized Legendre transform of a vector-function g, defined on a submahifold X is the following function $p = h(\alpha)$:*

$$h(\alpha) := \mathcal{L}(g)(\alpha) := \sum_{i=1}^{m} \alpha_{n-m+i} g_i(x(u(\alpha))) + \alpha' \cdot x(u(\alpha)). \tag{3.17}$$

The significance of GLT for the goals of this paper stems from the fact that GLT gives a convenient description of the singularities of the Radon transform Rf generated by the boundary of $\mathrm{supp}(f)$.

Theorem 3.4 *Let $D \subset \mathbf{R}^n$ be a compact domain bounded by a piecewise-C^1 surface S, f be a piecewise-smooth function such that $supp(f) = D$, $f|_D \in C^\infty(D)$, $f|_S \neq 0$, thus $S = singsupp(f)$. Denote $\hat{S} := singsupp(\hat{f})$. Then $\hat{S}$ is the image of S under the Generalized Legendre transform:*

$$\hat{S} = \mathcal{L}(S). \tag{3.18}$$

Proof. For a function f such that $\mathrm{supp}(f) = D$, $f|_D \in C^\infty(D)$, $f|_S \neq 0$ in a neighborhood of a point $\overline{y} \in S_{\{1,\ldots,m_j\}}$ one has:

$$WF(f) = N^* S_{\{1,\ldots,m_j\}}.$$

Thus the theorem is a consequence of Theorem 2.1 and Definition 3.1. □

Corollary 1 *Let $D \subset \mathbf{R}^n$ be a compact domain bounded by a piecewise-C^1 surface S, and χ_D is the characteristic function of D. Then*

$$singsupp(\hat{\chi}_D) = \mathcal{L}(S). \tag{3.19}$$

4 Volume invariant of intersecting surfaces

In this section we introduce a geometrical invariant characterizing oriented manifold, which is the intersection $S_{\{1,\dots,m\}}$ of m hypersurfaces $S_1,\dots,S_m$ intersecting at a point $\overline{y}\in S_{\{1,\dots,m\}}$. This invariant will play an important role in the next section, where we derive the asymptotics of the Radon transform near its singular support.

Let us assume that in a neighborhood of the point $\overline{y}$ the equation of S_i is $G_i(y)=0$, $i=1,\dots,m$. In order to describe the intrinsic geometry of $S_{\{1,\dots,m\}}$ let us assume that in a neighborhood of the point $\overline{y}$ the manifold $S_{\{1,\dots,m\}}$ is parametrized by the imbedding $F:\mathbf{R}^{n-m}\supset U\to\mathbf{R}^n$ such that $\overline{y}=F(0)$. Thus $S_{\{1,\dots,m\}}=\{y\,:\,y=F(u)\,|\,u\in U\}$ and $dF(0):T_0U\to T_{\overline{y}}S_{\{1,\dots,m\}}\subset T_{\overline{y}}\mathbf{R}^n$. Denote by $\mathbf{N}_j$, $j=1,\dots,m$, an exterior (with respect to the domain D) normal vector to the surface S_j considered as an element of $T_{\overline{y}}\mathbf{R}^n$ and by $Vol(\xi_1,\dots,\xi_n)$ the volume of the parallelepiped formed by the system of n vectors $\xi_1,\dots,\xi_n$, in $T_{\overline{y}}\mathbf{R}^n$, that is

$$Vol(\xi_1,\dots,\xi_n)=\det||\xi_i^j||_{i,j=1,\dots,n},$$

where ξ_i^j are the components of ξ_i in the standart orthonormal basis of $\mathbf{R}^n$.

Definition 4.1 *A frame $\mathbf{N}_1,\dots,\mathbf{N}_m,\eta_1,\dots,\eta_{n-m}$ in $T_{\overline{y}}\mathbf{R}^n$ is called an oriented frame (basis) if*

$$Vol(\mathbf{N}_1,\dots,\mathbf{N}_m,\eta_1,\dots,\eta_{n-m})>0. \tag{4.1}$$

Let $\overline{N}(\overline{y})\in T_{\overline{y}}\mathbf{R}^n$ be any unit normal to $S_{\{1,\dots,m\}}$ at the point $\overline{y}$. Denote by Q_1 the first fundamental form of the surface $S_{\{1,\dots,m\}}$ and by Q_2 the second fundamental form of

$S_{\{1,\ldots,m\}}$ in the direction $\overline{N}$ generated by this imbedding. The forms Q_1 and Q_2 at the point $\overline{y}$ are defined as follows:

$$Q_1(\overline{y}) := ||F'_{u_i}(\overline{y}) \cdot F'_{u_j}(\overline{y})||_{i,j=1,\ldots,n-m}, \tag{4.2}$$

$$Q_2(\overline{y}, \overline{N}(\overline{y})) := ||F''_{u_i u_j}(\overline{y}) \cdot \overline{N}(\overline{y})||_{i,j=1,\ldots,n-m}. \tag{4.3}$$

Denote also

$$|g(\overline{y})| := |\mathrm{det} Q_1(\overline{y})|, \quad K(\overline{y}, \overline{N}(\overline{y})) := |\mathrm{det} Q_2(\overline{y}, \overline{N}(\overline{y}))|. \tag{4.4}$$

Let us consider the vectors $\nabla G_j(\overline{y}) = (\partial G_j(\overline{y})/\partial y_1, \ldots, \partial G_j(\overline{y})/\partial y_n)$, $j = 1, \ldots, m$, as elements of $T_{\overline{y}}\mathbf{R}^n$. We say that equations $G_j(y) = 0$ and a parametrization $y = F(u)$ are consistent if the frame $\{\nabla G_1(\overline{y}), \ldots, \nabla G_m(\overline{y}), dF(0)\frac{\partial}{\partial u_1}, \ldots, dF(0)\frac{\partial}{\partial u_m}\}$ is oriented, where

$$dF\frac{\partial}{\partial u_j} = \sum_{l=1}^{n} \frac{\partial F_l}{\partial u_j}\frac{\partial}{\partial y_l}.$$

Below we assume that the equations of surfaces and the parametrization of the intersection surface are consistent.

Further, one has the following decomposition:

$$\overline{N}(\overline{y}) = \sum_{i=1}^{m} \lambda_i \nabla G_i(\overline{y}). \tag{4.5}$$

Definition 4.2 *We call the volume invariant of intersecting hypersurfaces $S_1, \ldots, S_m$ at a point $\overline{y}$ in the direction $\overline{N}(\overline{y})$ the following expression:*

$$v(\overline{y}, \overline{N}(\overline{y}), S_1, \ldots, S_m) :=$$

$$Vol(\lambda_1 \nabla G_1(\overline{y}), \ldots, \lambda_m \nabla G_m(\overline{y}), dF(\overline{y})\frac{\partial}{\partial u_1}, \ldots, dF(\overline{y})\frac{\partial}{\partial u_m})|g(\overline{y})|^{-1} K(\overline{y}, \overline{N}(\overline{y}))^{1/2} \tag{4.6}$$

Theorem 4.3 *The volume invariant $v(\overline{y}, \overline{N}(\overline{y}), S_1, \ldots, S_m)$ depends neither on the choice of the equations $G_i(y) = 0$, $i = 1, \ldots, m$, nor on the parametrization of $S_{\{1,\ldots,m\}}$ provided that they are consistent.*

Proof. To prove this theorem note that the change of an equation $G_i(y) = 0$ results in the multiplication of $\nabla G_i(\overline{y})$ by some scalar and does not affect the vector $\lambda_i \nabla G_i(\overline{y})$ in (4.5).

A reparametrization $u' = \Phi(u)$ of $S_{\{1,\ldots,m\}}$ keeping the orientation (here the consistency condition is used) leads to the multiplication of $Vol(\lambda_1 \nabla G_1(\overline{y}), \ldots, \lambda_m \nabla G_m(\overline{y}), dF(\overline{y})\frac{\partial}{\partial u_1}, \ldots, dF(\overline{y})\frac{\partial}{\partial u_m})$ by the Jacobian $J = \det||\frac{\partial \Psi_i}{\partial u_j}(0)||_{i,j=1,\ldots,n-m} > 0$ and to the multiplication of $|g(\overline{y})|$ by J^2. Thus

$$F''_{u_i u_j}(\overline{y}) \cdot \overline{N}(\overline{y}) = \sum_{k,l=1}^{n-m} F''_{u'_k u'_l}(\overline{y}) \frac{\partial \Psi_i}{\partial u_j}(0) \frac{\partial \Psi_i}{\partial u_j}(0) \cdot \overline{N}(\overline{y}) + F'_{u'_k} \frac{\partial^2 \Psi_k}{\partial u_i \partial u_j}(0) \cdot \overline{N}(\overline{y}).$$

Because $\overline{N}$ is normal to $S_{\{1,\ldots,m\}}$ one has

$$F'_{u'_k} \cdot \overline{N}(\overline{y}) = 0.$$

Therefore the reparametrization leads to the multiplication of $K(\overline{y}, \overline{N}(\overline{y}))$ by J^2, and the conclusion of the theorem follows from the definition of $v(\overline{y}, \overline{N}(\overline{y}), S_1, \ldots, S_m)$. $\square$

Now let us calculate $v(\overline{y}, \overline{N}(\overline{y}), S_1, \ldots, S_m)$ in a special coordinate system, where $G_i = y_i - g_i(y')$, $u = y' = (y_{m+1}, \ldots, y_n)$. In this coordinate system one gets:

$$\nabla G_i(y) = \left(0, \ldots, 1, \ldots, 0, -\frac{\partial g_i}{\partial y_{m+1}}, \ldots, -\frac{\partial g_i}{\partial y_n}\right), \tag{4.7}$$

where the unity appears at the place i, and $\lambda_i = \overline{\alpha}_i$ for $i = 1, \ldots, m$. Since

$$F(y') = (g_1(y'), \ldots, g_m(y'), y_{m+1}, \ldots, y_n),$$

and

$$dF\frac{\partial}{\partial y_{m+i}} = \sum_{l=1}^{m} \frac{\partial g_l}{\partial y_{m+i}} \frac{\partial}{\partial y_l} + \frac{\partial}{\partial y_{m+i}},$$

one gets

$$Vol(\nabla G_1(\overline{y}), \ldots, \nabla G_m(\overline{y}), dF(\overline{y})\frac{\partial}{\partial u_1}, \ldots, dF(\overline{y})\frac{\partial}{\partial u_m}) =$$

$$= \det \begin{pmatrix} E_m & ||\partial g_i/\partial y_{m+j}||^t \\ -||\partial g_i/\partial y_{m+j}|| & E_{n-m} \end{pmatrix} =$$

$$= \det\left(E_{n-m} + ||\partial g_i/\partial y_{m+j}||^t \cdot ||\partial g_i/\partial y_{m+j}|| \right),$$

where E_l is the unit $l \times l$ matrix and A^t stands for the transposed matrix.

Since

$$Q_1 = E_{n-m} + ||\sum_{l=1}^{m} \partial g_l/\partial y_{m+i} \partial g_l/\partial y_{m+j}||_{i,j=1,\ldots,n-m},$$

one gets

$$\det\left(E_{n-m} + ||\partial g_i/\partial y_{m+j}||^t \cdot ||\partial g_i/\partial y_{m+j}|| \right) = |g(\overline{y})| > 0,$$

and the consistency condition holds.

Therefore

$$Vol(\lambda_1 \nabla G_1(\overline{y}), \ldots, \lambda_m \nabla G_m(\overline{y}), dF(\overline{y})\frac{\partial}{\partial u_1}, \ldots, dF(\overline{y})\frac{\partial}{\partial u_m}) = \lambda_1 \cdot \ldots \cdot \lambda_m |g(\overline{y})|, \tag{4.8}$$

where $\lambda_1, \ldots, \lambda_m$ are the same as in formula (4.6). Denote by $N_1, \ldots, N_n$ the components of the vector $\mathbf{N}(\overline{y})$, then

$$|K(\overline{y}, \overline{N}(\overline{y}))| = \left| \det \left\| \sum_{l=1}^{m} \mathbf{N}_l \frac{\partial^2 g_l}{\partial y_{m+i} \partial y_{m+j}} (\overline{y}) \right\|_{i,j=1,\ldots,n-m} \right|.$$

Therefore one obtains a formula for v::

$$v(\overline{y}, \overline{N}(\overline{y}), S_1, \ldots, S_m) = \lambda_1 \cdot \ldots \cdot \lambda_m \cdot \left| \det \left\| \sum_{l=1}^{m} \mathbf{N}_l \frac{\partial^2 g_l}{\partial y_{m+i} \partial y_{m+j}} (\overline{y}) \right\|_{i,j=1,\ldots,n-m} \right|^{1/2}. \tag{4.9}$$

This representation of the volume invariant in a special coordinate system is used in the next section for deriving (5.18).

5 Asymptotics by smoothness of the Radon transform

In this section we study the behavior of the singular part of $\hat{f}(\overline{\alpha}, p)$, when p approaches $\overline{p}$ and $(\overline{\alpha}, \overline{p})$ belongs to singsupp$(\hat{f}) := \hat{S}$. This problem has been first studied in [5], [6]. We give a different approach and obtain more general results: the full asymptotics of $\hat{f}(\alpha, p)$ near $\hat{S}$ (in [5], [6] essentially the first terms of the asymptotic expansion was obtained). As in the previous sections we assume that $D \subset \mathbf{R}^n$ is a compact domain bounded by a surface S, which consists of a finite number of subsurfaces of various codimensions (a stratified surface), and f is a piecewise-C^N function such that supp$(f) = D$ and $f|_D \in C^N(D)$. Singularities of $\hat{f}(\overline{\alpha}, p)$ are generated by singularities of $f(x)$ inside D and by the boundary S. Below we assume that f is smooth in D, $f = 0$ in $D' = \mathbf{R}^n \setminus \bar{D}$, $\bar{D}$ is the closure of D, $f|_S \neq 0$, and study singularities of $\hat{f}(\overline{\alpha}, p)$ generated by the boundary. Our method allows one to study the case when f vanishes on S up to a finite order.

If $(\overline{\alpha}, \overline{p}) \in$ singsupp$(\hat{f})$ then $\mathcal{L}^{-1}(\overline{\alpha}, \overline{p})$ is a nonempty subset of S. We assume that $\mathcal{L}^{-1}(\overline{\alpha}, \overline{p})$ is a finite set, so

$$\mathcal{L}^{-1}(\overline{\alpha}, \overline{p}) = \{\overline{y}_j \, : \, j = 1, \dots, j_0, \quad \overline{y}_j \in S^j_{\{1,\dots,m_j\}}\},$$

and $S^j_{\{1,\dots,m_j\}} = \cap_{i=1}^{m_j} S^j_i$, where S^j_i is the i-th surface passing through the point $\overline{y}_j \in S$. The index j here and below corresponds to a point $\overline{y}_j \in S_{\{1,\dots,m_j\}}$ at which the plane defined by the point $(\alpha, \overline{p})$ is tangent to S.

Denote by I_j and σ_j correspondingly the index, that is, the number of negative eigenvalues and the signature, that is, the number of positive eigenvalues minus the number of negative eigenvalues of the second fundamental form in the direction $\mathbf{N} = \overline{\alpha}/|\overline{\alpha}|$ of a surface $S^j_{\{1,\ldots,m_j\}}$ at the point $\overline{y}_j$, and by v_j the volume invariant of intersecting hypersurfaces $S^j_1, \ldots, S^j_{m_j}$ at the point $\overline{y}_j$ in the direction $\mathbf{N}$, $|\alpha|$ denotes the length of the vector α.

Let us introduce the following functions

$$s_1(j,p) = p_+^{\frac{n+m_j}{2}-1}, \tag{5.1}$$

$$s_2(j,p) = \begin{cases} p^{\frac{n+m_j}{2}-1}\log|p|, & \text{if } n+m_j \text{ is even,} \\ p_-^{\frac{n+m_j}{2}-1}, & \text{if } n+m_j \text{ is odd,} \end{cases} \tag{5.2}$$

where $p_+ := \max\{0,p\}$, $p_- := \max\{0,-p\}$.

Denote by $[a]$ the integer part of a real number a.

Theorem 5.1 *If $K(\overline{y}_j, \overline{\alpha}/|\overline{\alpha}|) \neq 0$ for $j = 1,\ldots,j_0$, then there exist polynomials*

$$P_{j,i}(z) = \sum_{l=0}^{N-\left[\frac{n+m_j}{2}-1\right]} a_{j,i,l} z^l, \quad j = 1,\ldots,j_0, \quad i = 1,2,$$

and a neighborhood $U_{\overline{p}}$ of point $\overline{p}$ such that:

I.

$$\hat{f}(\overline{\alpha},p) = \sum_{j=1}^{j_0} (P_{j,1}(p-\overline{p})s_1(j,p-\overline{p}) + P_{j,2}(p-\overline{p})s_2(j,p-\overline{p})) \quad (mod\ C^{N_1}(U_{\overline{p}})), \tag{5.3}$$

where $N_1 = [(N+n+\bar{m})/2]$, $\bar{m} = \min\{m_j\}$.

II. *1. If $n+m_j$ is even and I_j is even then*

$$a_{j,1,0} = (2\pi)^{\frac{n-m_j}{2}} i^{m_j} \frac{(-1)^{[(n+m_j)/4]}}{((n+m_j)/2-1)!} \frac{|\alpha|^{1-\frac{n+m_j}{2}}}{v_j} e^{-\pi i \sigma_j/4} f(\overline{y}_j), \tag{5.4}$$

$$a_{j,2,0} = 0, \tag{5.5}$$

2. If $n + m_j$ is even and I_j is odd then

$$a_{j,1,0} = 0, \tag{5.6}$$

$$a_{j,2,0} = 2(2\pi)^{\frac{n-m_j}{2}-1} i^{m_j} \frac{(-1)^{[(n+m_j)/4]+1}}{\left(\frac{n+m_j}{2}-1\right)!} \frac{|\alpha|^{1-\frac{n+m_j}{2}}}{v_j} e^{-\pi i \sigma_j/4} f(\overline{y}_j), \tag{5.7}$$

3. If $n + m_j$ is odd and I_j is even then

$$a_{j,1,0} = 2(2\pi)^{\frac{n-m_j}{2}-1} i^{m_j} e^{-\frac{\pi i}{4}(n+m_j-2)} \Gamma\left(1 - \frac{n+m_j}{2}\right) \frac{|\alpha|^{1-\frac{n+m_j}{2}}}{v_j} e^{-\pi i \sigma_j/4} f(\overline{y}'), \tag{5.8}$$

$$a_{j,2,0} = 0, \tag{5.9}$$

4. If $n + m_j$ is odd and I_j is odd then

$$a_{j,1,0} = 0, \tag{5.10}$$

$$a_{j,2,0} = 2(2\pi)^{\frac{n-m_j}{2}-1} i^{m_j} e^{\frac{\pi i}{4}(n+m_j-2)} \Gamma\left(1 - \frac{n+m_j}{2}\right) \frac{|\alpha|^{1-\frac{n+m_j}{2}}}{v_j} e^{-\pi i \sigma_j/4} f(\overline{y}'). \tag{5.11}$$

Proof. Denote by U_j a small neighborhood of $\overline{y}_j$ and by $\chi_j(y) \in C_0^\infty(\mathbf{R}^n)$ a cut-off function with support in U_j and equal to one in a neighborhood of $\overline{y}_j$. Then

$$\hat{f}(\overline{\alpha}, p) = \frac{1}{2\pi} \sum_{j=0}^{j_0} \hat{f}_j(\overline{\alpha}, p),$$

where

$$\hat{f}_j(\overline{\alpha}, p) = |\alpha| \int\limits_{\mathbf{R}^n} \int\limits_{\mathbf{R}} e^{i\theta(\overline{p} - \overline{\alpha}\cdot y)} e^{i\theta(p-\overline{p})} \chi_j(y) f(y) dy d\theta, \quad j \geq 1, \tag{5.12}$$

and

$$\hat{f}_0(\overline{\alpha}, p) = |\alpha| \int\limits_{\mathbf{R}^n} \int\limits_{\mathbf{R}} e^{i\theta(\overline{p} - \overline{\alpha}\cdot y)} e^{i\theta(p-\overline{p})} \left[1 - \sum_{j=1}^{j_0} \chi(y)\right] f(y) dy d\theta.$$

Since $\left[1-\sum_{j=1}^{j_0}\chi(y)\right]f(y)$ is zero in some neighborhoods of the points $\overline{y}_j$, one concludes that $(\overline{\alpha},\overline{p})\notin \mathrm{singsupp}(\hat{f}_0)$.

In what follows we study the behavior of the singular part of $\hat{f}_j(\overline{\alpha},p)$ for a fixed j and omit subindex j by this reason. Thus let

$$\overline{y}\in S_{\{1,\dots,m\}},\quad D\cap U=\{y\,|\,y\in U,\quad G_k(y)\le 0,\quad k=1,\dots,m\},$$

$$\overline{\alpha}\in \mathrm{span}(\nabla G_1(\overline{y}),\dots,\nabla G_m(\overline{y})),\quad \overline{p}=\overline{y}\cdot\overline{\alpha}.$$

From (5.12) one gets:

$$\hat{f}(\overline{\alpha},p)=\frac{1}{2\pi}\left(I(\overline{\alpha},p)+\tilde{I}(\overline{\alpha},p)\right), \tag{5.13}$$

where

$$I(\overline{\alpha},p)=|\alpha|\int\limits_{\mathbf{R}^n}dy\int\limits_{|\theta|\ge 1}d\theta e^{i\theta(\overline{p}-\overline{\alpha}\cdot y)}e^{i\theta(p-\overline{p})}\chi(y)f(y),$$

and

$$\tilde{I}(\overline{\alpha},p)=|\alpha|\int\limits_{\mathbf{R}^n}dy\int\limits_{|\theta|\le 1}d\theta e^{i\theta(\overline{p}-\overline{\alpha}\cdot y)}e^{i\theta(p-\overline{p})}\chi(y)f(y)$$

is a C^∞-function of p.

Let us choose local coordinates y such that $G_k(y)=y_k-g_k(y')$, $k=1,\dots,m$, $y'=(y_{m+1},\dots,y_n)$ as in the previous section. Then

$$I(\overline{\alpha},p)=c(\overline{\alpha})\int\limits_{|\theta|\ge 1}e^{i\theta(p-\overline{p})}\theta^{-m}d\theta\int\limits_{\mathbf{R}^n}dy\chi(y)f(y)\partial_{y_1}\dots\partial_{y_m}(\exp[i\theta(\overline{p}-\overline{\alpha}\cdot y)]),$$

where

$$c(\overline{\alpha})=i^m\frac{|\alpha|}{\overline{\alpha}_1\cdot\ldots\cdot\overline{\alpha}_m}. \tag{5.14}$$

Integrating by parts with respect to $y_1, \dots, y_m$, one gets:

$$\int\limits_{\mathbf{R}^n} dy\chi(y)f(y)\partial_{y_1}\dots\partial_{y_m}(\exp[i\theta(\overline{p}-\overline{\alpha}\cdot y)]) = \sum_{k=0}^{N}\theta^{-j}\int\limits_{\mathbf{R}^{n-m}} e^{i\theta\psi(y')}P_k(\partial_{y'})[\chi(y')f(y')]dy' + |\theta|^{-N-1}R(\theta),$$

where $\chi(y') = \chi(g_1(y'), \dots, g_m(y'), y')$, $f(y') = f(g_1(y'), \dots, g_m(y'), y')$, $P_0 = 1$, $P_k(\partial_{y'})$ is a partial differential operator of order k with coefficients depending only on $\overline{\alpha}$,

$$\psi(y') := \sum_{k=1}^{m}\overline{\alpha}_k g_k(y') + \sum_{k=m+1}^{n}\overline{\alpha}_k y_k - \overline{p},$$

and $R(\theta) \in C((-\infty, -1] \cup [1, \infty))$ is a bounded function.

Thus

$$I(\overline{\alpha}, p) = \sum_{k=0}^{N} I_k(\overline{\alpha}, p) + I_{N+1}(\overline{\alpha}, p),$$

where

$$I_k(\overline{\alpha}, p) = c(\overline{\alpha})\int\limits_{|\theta|\geq 1} e^{i\theta(p-\overline{p})}\theta^{-m-k}d\theta\int\limits_{\mathbf{R}^{n-m}} e^{-i\theta\psi(y')}P_k(\partial_{y'})[\chi(y')f(y')]dy',$$

for $k = 0, 1, \dots, N$ and

$$I_{N+1}(\overline{\alpha}, p) = c(\overline{\alpha})\int\limits_{|\theta|\geq 1} e^{i\theta(p-\overline{p})}\theta^{-m}|\theta|^{-N-1}R(\theta)d\theta \in C^{m+N}(\mathbf{R}).$$

Denote by $H(y')$ the Hessian of $\psi(y')$.

Lemma 5.2 *The point $y' = \overline{y}'$ is a critical point for $\psi(y')$ and*

$$H(\overline{y}') = |\overline{\alpha}|^{n-m}Q_2(\overline{y}, \overline{\alpha}/|\overline{\alpha}|), \tag{5.15}$$

where Q_2 is the second fundamental form of the surface $S_{\{1,\dots,m\}}$ in the direction the $\overline{\alpha}/|\overline{\alpha}|$ at the point $\overline{y}$ (see (4.3)).

Proof. Since $(\overline{\alpha},\overline{p}) \in \text{singsupp}\hat{f}$, the hyperplane defined by the point $(\overline{\alpha},\overline{p})$ is tangent to $S_{\{1,\ldots,m\}}$ at the point $\overline{y}$ and there exist constants $\lambda_1,\ldots,\lambda_m$ such that $\mathbf{N}(\overline{y}) := \overline{\alpha}/|\overline{\alpha}| = \sum_{k=1}^m \lambda_k \nabla G_k(\overline{y})$.

Therefore (cf (4.7))

$$\overline{\alpha}_k = |\overline{\alpha}|\lambda_k \text{ for } k = 1,\ldots,m, \tag{5.16}$$

and

$$\overline{\alpha}_k = -|\overline{\alpha}|\sum_{i=1}^m \lambda_i \frac{\partial g_i}{\partial y_k}\bigg|_{y=\overline{y}'} = -\sum_{i=1}^m \overline{\alpha}_i \frac{\partial g_i}{\partial y_k}\bigg|_{y=\overline{y}'} \text{ for } k = m+1,\ldots,n.$$

Thus for $i,k = m+1,\ldots,n$, one has

$$\frac{\partial \psi}{\partial y_i}\bigg|_{y=\overline{y}'} = \sum_{l=1}^m \overline{\alpha}_l \frac{\partial g_l}{\partial y_i}\bigg|_{y=\overline{y}'} + \overline{\alpha}_i = 0,$$

and

$$\frac{\partial^2 \psi}{\partial y_i \partial y_k} = \sum_{l=1}^m \overline{\alpha}_l \frac{\partial^2 g_l}{\partial y_i \partial y_k} = |\overline{\alpha}|\sum_{l=1}^m \mathbf{N}_l \frac{\partial^2 g_l}{\partial y_i \partial y_k},$$

where

$$\left\|\mathbf{N}_l \frac{\partial^2 g_l}{\partial y_i \partial y_k}\right\|_{i,k=1,\ldots,n-m} = Q_2(\overline{y},\mathbf{N}(\overline{y}))$$

is the second fundamental form of the surface $S_{\{1,\ldots,m\}}$ in the direction $\mathbf{N}(\overline{y}) = \overline{\alpha}/|\overline{\alpha}|$. $\square$

Denote $F_k(y') := P_k(\partial_{y'})[\chi(y')f(y')]$, thus $F_k(y') \in C^{N-k}(\mathbf{R}^{n-m})$. By the assumption of Theorem 5.1 it follows from Lemma 5.2 that the Hessian $H(\overline{y})$ is nonsingular. Let I be a number of the negative eigenvalues of $H(\overline{y})$. It follows from the stationary phase formula ([7, p. 431]) that

$$\int\limits_{\mathbf{R}^{n-m}} e^{-i\theta\psi(y')}F_k(y')dy' = |\theta|^{-(n-m)/2}\sum_{l=0}^{[(N-k)/2]} C_{k,l}^{\pm}|\theta|^{-l} + O(|\theta|^{-N_k-1}), \quad \theta \to \pm\infty,$$

where $C^{\pm}_{k,l}$ are constants depending on $D^\mu f(\overline{y})$, $|\mu| \leq k+2l$, $C^{-}_{k,l}$ is the complex conjugate of $C^{+}_{k,l}$, $N_k := \frac{n-m}{2} + \left[\frac{N-k}{2}\right]$ and the principal term of the asymptotics is defined by the coefficient

$$C^{\pm}_{k,0} = (2\pi)^{(n-m)/2} e^{\mp\pi i(n-m-2I)/4} \frac{F_j(\overline{y}')}{\sqrt{|\mathrm{det} H(\overline{y}')|}}.$$

Thus

$$I(\overline{\alpha}, p) = c(\overline{\alpha}) \sum_{k=0}^{N} \sum_{l=0}^{N-k} \int_1^\infty \left(C^{+}_{k,l} e^{i\theta(p-\overline{p})} + (-1)^{m+k} C^{-}_{k,l} e^{-i\theta(p-\overline{p})} \right) \theta^{-\frac{n+m}{2}-k-l} d\theta \pmod{C^{N_1}(\mathbf{R})},$$

where $N_1 = [(N+n+m)/2]$ and $c(\overline{\alpha})$ is defined in (5.14). For even $n+m$ we present the last sum in the form:

$$I(\overline{\alpha}, p) = C_0 J_0(\overline{\alpha}, p) + \sum_{k=1}^{N} (C^{+}_k J^{+}_k(\overline{\alpha}, p) + C^{-}_k J^{-}_k(\overline{\alpha}, p)) \pmod{C^{N_1}(\mathbf{R})},$$

and for odd $n+m$ in the form:

$$I(\overline{\alpha}, p) = C_0 J_0(\overline{\alpha}, p) + \sum_{j=1}^{N} (\tilde{C}^{+}_j \tilde{J}^{+}_j(\overline{\alpha}, p) + \tilde{C}^{-}_j \tilde{J}^{-}_j(\overline{\alpha}, p)) \pmod{C^{N_1}(\mathbf{R})},$$

where

$$C_0 = (2\pi)^{(n-m)/2} i^m \frac{|\alpha|}{\overline{\alpha}_1 \cdot \ldots \cdot \overline{\alpha}_m} e^{-\pi i(n-m-2I)/4} \frac{f(\overline{y})}{\sqrt{|\mathrm{det} H(\overline{y}')|}}, \tag{5.17}$$

$$J_0(\overline{\alpha}, p) = \int_1^\infty (e^{i\theta(p-\overline{p})} + (-1)^{-m} e^{\pi i(n-m-2I)/2} e^{-i\theta(p-\overline{p})}) \theta^{-(n+m)/2} d\theta,$$

and for $k = 1, \ldots, N$, one has:

$$J^{+}_k(\overline{\alpha}, p) = \int_1^\infty \cos(\theta(p-\overline{p})) \theta^{-\frac{n+m}{2}-k} d\theta,$$

$$J^{-}_k(\overline{\alpha}, p) = \int_1^\infty \sin(\theta(p-\overline{p})) \theta^{-\frac{n+m}{2}-k} d\theta,$$

$$\tilde{J}_k^{\pm}(\overline{\alpha}, p) = \int_1^\infty e^{\pm i\theta(p-\overline{p})}\theta^{-\frac{n+m}{2}-k}d\theta,$$

$$C_k^+ = c(\overline{\alpha})\sum_{l=0}^{k}\left[C_{l,k-l}^+ + (-1)^{m+l}C_{l,k-l}^-\right], \quad C_k^- = ic(\overline{\alpha})\sum_{l=0}^{k}\left[C_{l,k-l}^+ - (-1)^{m+l}C_{l,k-l}^-\right],$$

$$\tilde{C}_k^+ = c(\overline{\alpha})\sum_{l=0}^{k}C_{l,k-l}^+, \quad \tilde{C}_k^- = c(\overline{\alpha})\sum_{l=0}^{k}(-1)^{m+k}C_{l,k-l}^-,$$

where $c(\overline{\alpha})$ is defined in (5.14). It follows from (5.16), (4.9) and (5.15) that

$$\overline{\alpha}_1\cdot\ldots\cdot\overline{\alpha}_m = |\overline{\alpha}|^m\lambda_1\cdot\ldots\cdot\lambda_m,$$

and so

$$\overline{\alpha}_1\cdot\ldots\cdot\overline{\alpha}_m|H(\overline{y}')|^{1/2} = |\overline{\alpha}|^{(n+m)/2}v(\overline{y}, \overline{\alpha}/|\overline{\alpha}|, S_1, \ldots, S_m). \tag{5.18}$$

Thus from (5.17) one gets that

$$C_0 = \frac{(2\pi)^{(n-m)/2}i^m e^{-\pi i(n-m-2I)/4}}{|\overline{\alpha}|^{(n+m)/2}v(\overline{y}, \overline{\alpha}/|\overline{\alpha}|, S_1, \ldots, S_m)}f(\overline{y}). \tag{5.19}$$

Claim: For even $n+m$ the statement of Theorem 5.1 follows from Lemma 6.1 (Sect. 6 below).

Let us prove this claim. Indeed, in this case

$$J_0(\overline{\alpha}, p) = \int_1^\infty \left[e^{i\theta(p-\overline{p})} + (-1)^{\frac{n+m}{2}-I}e^{-i\theta(p-\overline{p})}\right]\theta^{-(n+m)/2}d\theta.$$

If $\frac{n+m}{2} - I$ is even, then

$$J_0(\overline{\alpha}, p) = 2\int_1^\infty \cos(\theta(p-\overline{p}))\theta^{-(n+m)/2}d\theta.$$

If $\frac{n+m}{2} - I$ is even and I is even, then $(n+m)/2$ is even and, from Section 6, formula (6.3), one gets

$$J_0(\overline{\alpha}, p) = 2\pi\frac{(-1)^{(n+m)/4}}{((n+m)/2-1)!}(p-\overline{p})_+^{\frac{n+m}{2}-1} \pmod{C^\infty(\mathbf{R})}.$$

If I is odd, then $(n+m)/2$ is odd and, from (6.1), one gets

$$J_0(\overline{\alpha},p)=2\frac{(-1)^{(n+m+2)/4}}{((n+m)/2-1)!}(p-\overline{p})^{\frac{n+m}{2}-1}\log|p-\overline{p}| \quad (\mathrm{mod}\ C^\infty(\mathbf{R})).$$

Similarly, if $(n+m)/2-I$ is odd, then

$$J_0(\overline{\alpha},p)=2\int\limits_1^\infty \sin(\theta(p-\overline{p}))\theta^{-(n+m)/2}d\theta.$$

Therefore, for even I, from formula (6.2), one gets

$$J_0(\overline{\alpha},p)=2\pi\frac{(-1)^{(n+m-2)/4}}{((n+m)/2-1)!}(p-\overline{p})_+^{\frac{n+m}{2}-1} \quad (\mathrm{mod}\ C^\infty(\mathbf{R})),$$

and for odd I, from formula (6.4), one obtains

$$J_0(\overline{\alpha},p)=2\frac{(-1)^{(n+m)/4}}{((n+m)/2-1)!}(p-\overline{p})^{\frac{n+m}{2}-1}\log|p-\overline{p}| \quad (\mathrm{mod}\ C^\infty(\mathbf{R})).$$

Then for $k=1,\dots,N$, it follows from Lemma 6.1 (see Sect. 6 below) that

$$C_k^+J_k^+(\overline{\alpha},p)+C_k^-J_k^-(\overline{\alpha},p)=(p-\overline{p})^k(c_{k,1}(p-\overline{p})_+^{\frac{n+m}{2}-1}+c_{k,2}(p-\overline{p})^{\frac{n+m}{2}-1}\log|p-\overline{p}|) \quad (\mathrm{mod}\ C^\infty(\mathbf{R})),$$

with constants $c_{k,1}$, $c_{k,2}$ depending on $C_k^\pm$.

For odd $n+m$ the statement of Theorem 5.1 follows from Lemma 6.2. In this case

$$J_0(\overline{\alpha},p)=\int\limits_1^\infty e^{i\theta(p-\overline{p})}\theta^{-(n+m)/2}d\theta+e^{\pi i(n+m-2I)/2}\int\limits_1^\infty e^{-i\theta(p-\overline{p})}\theta^{-(n+m)/2}d\theta=$$

$$=\Gamma\left(\frac{1}{2}-k\right)\left(e^{\mp i\pi(n+m-2)/4}(p-\overline{p})_+^{\frac{n+m}{2}-1}+e^{\pm i\pi(n+m-2)/4}(p-\overline{p})_-^{\frac{n+m}{2}-1}\right) \quad (\mathrm{mod}\ C^\infty(\mathbf{R})).$$

Thus

$$J_0(\overline{\alpha},p)=\Gamma\left(\frac{1}{2}-k\right)\left(e^{-i\pi(n+m-2)/4}(p-\overline{p})_+^{\frac{n+m}{2}-1}(1+(-1)^{n+m-I-1})+\right.$$

$$+e^{i\pi(n+m-2)/4}(p-\overline{p})_{-}^{\frac{n+m}{2}-1}(1+(-1)^{I-1})\Big)J_0(\overline{\alpha},p),$$

and for $k=1,\ldots,N$

$$\tilde{C}_k^{+}\tilde{J}_k^{+}(\overline{\alpha},p)+\tilde{C}_k^{-}\tilde{J}_k^{-}(\overline{\alpha},p)=\tilde{c}_{k,1}(p-\overline{p})_{+}^{\frac{n+m}{2}+k-1}+\tilde{c}_{k,2}(p-\overline{p})_{-}^{\frac{n+m}{2}+k-1}\quad (\mathrm{mod}\ C^{\infty}(\mathbf{R})).$$

□

6 Appendix

Lemma 6.1 *Let x be a real number and k be a nonnegative integer. Then:*

1. *For even k*

$$\int_1^{\infty}\cos(x\theta)\theta^{-k-1}d\theta=\frac{(-1)^{\frac{k}{2}+1}}{k!}x^k\log|x|\quad (mod\ C^{\infty}(\mathbf{R})), \tag{6.1}$$

$$\int_1^{\infty}\sin(x\theta)\theta^{-k-1}d\theta=\pi\frac{(-1)^{k/2}}{k!}x_{+}^{k}\quad (mod\ C^{\infty}(\mathbf{R})), \tag{6.2}$$

2. *For odd k*

$$\int_1^{\infty}\cos(x\theta)\theta^{-k-1}d\theta=\pi\frac{(-1)^{(k+1)/2}}{k!}x_{+}^{k}\quad (mod\ C^{\infty}(\mathbf{R})), \tag{6.3}$$

$$\int_1^{\infty}\sin(x\theta)\theta^{-k-1}d\theta=\frac{(-1)^{(k+1)/2}}{k!}x^k\log|x|\quad (mod\ C^{\infty}(\mathbf{R})), \tag{6.4}$$

Proof. For even k integrating by parts one gets

$$\int_1^{\infty}\cos(x\theta)\theta^{-k-1}d\theta=Q_1(x)\cos x+Q_2(x)\sin x+\frac{(-1)^{k/2}}{k!}x^k\int_1^{\infty}\cos(x\theta)\theta^{-1}d\theta, \tag{6.5}$$

$$\int_1^{\infty}\sin(x\theta)\theta^{-k-1}d\theta=Q_3(x)\cos x+Q_4(x)\sin x+\frac{(-1)^{k/2}}{k!}x^k\int_1^{\infty}\sin(x\theta)\theta^{-1}d\theta, \tag{6.6}$$

where $Q_l(x)$, $l = 1, \dots, 4$, are polynomials.

For positive x the integrals in the right-hand sides of this formulas can be expressed by means of the following integrals (see [1]):

$$\mathrm{Ci}(x) := -\int_x^\infty \frac{\cos t}{t} dt, \quad \mathrm{si}(x) := -\int_x^\infty \frac{\sin t}{t} dt.$$

From (6.5) one gets:

$$\int_1^\infty \cos(x\theta)\theta^{-k-1} d\theta = \frac{(-1)^{\frac{k}{2}+1}}{k!} x^k \mathrm{Ci}(|x|), \quad (\mathrm{mod}\ C^\infty(\mathbf{R})),$$

and from (6.6) one obtains:

$$\int_1^\infty \sin(x\theta)\theta^{-k-1} d\theta = \frac{(-1)^{\frac{k}{2}+1}}{k!} x^k \mathrm{sign}(x)\mathrm{si}(|x|) \quad (\mathrm{mod}\ C^\infty(\mathbf{R})).$$

For positive x one has (see [1]):

$$\mathrm{Ci}(x) = \gamma + \log x + \sum_{j=0}^\infty \frac{(-1)^j x^{2j}}{(2j)!(2j)}, \quad \mathrm{si}(x) = -\frac{\pi}{2} + \sum_{j=0}^\infty \frac{(-1)^j x^{2j+1}}{(2j+1)!(2j+1)},$$

where $\gamma > 0$ is the Euler constant. Therefore, modulo $C^\infty(\mathbf{R})$ functions, one has

$$\mathrm{Ci}(|x|) = \log|x|, \quad \mathrm{sign}(x)\mathrm{si}(|x|) = -\frac{\pi}{2}\mathrm{sign}(x), \quad (\mathrm{mod}\ C^\infty(\mathbf{R})).$$

For nonnegative integer k one has:

$$x^k \mathrm{sign}(x) = 2x_+^k = -2x_-^k \quad (\mathrm{mod}\ C^\infty(\mathbf{R})).$$

Indeed, if k is even, then $x^k\mathrm{sign}(x) = |x|^k\mathrm{sign}(x) = x_+^k - x_-^k$ and $x^k = |x|^k = x_+^k + x_-^k$ (see [3], p. 50). For odd k, $x^k\mathrm{sign}(x) = |x|^k = x_+^k + x_-^k$ and $x^k = |x|^k\mathrm{sign}(x) = x_+^k - x_-^k$. Thus (6.1) and (6.2) are proved. Similarly (6.3) and (6.4) can be derived from the following formulas for odd k:

$$\int_1^\infty \cos(x\theta)\theta^{-k-1}d\theta = \frac{(-1)^{(k+1)/2}}{k!}x^k \int_1^\infty \sin(x\theta)\theta^{-1}d\theta, \quad (\mathrm{mod}\ C^\infty(\mathbf{R})),$$

and

$$\int_1^\infty \sin(x\theta)\theta^{-k-1}d\theta = -\frac{(-1)^{(k+1)/2}}{k!}x^k \int_1^\infty \cos(x\theta)\theta^{-1}d\theta, \quad (\mathrm{mod}\ C^\infty(\mathbf{R})).$$

□

Lemma 6.2 *For $x \in \mathbf{R}\setminus\{0\}$ and nonnegative integer k one has:*

$$\int_1^\infty e^{\pm i\theta x}\theta^{-k-1/2}d\theta =$$

$$= \Gamma(1/2-k)\left(e^{\mp i\pi(2k-1)/4}x_+^{k-1/2} + e^{\pm i\pi(2k-1)/4}x_-^{k-1/2}\right) \quad (mod\ C^\infty(\mathbf{R})). \qquad (6.7)$$

Proof. By the definition of the incomplete Gamma-function:

$$\Gamma(\nu,z) := \int_\nu^\infty e^{-t}t^{\nu-1}dt,$$

one has:

$$\int_1^\infty e^{\pm i\theta x}\theta^{-k-1/2}d\theta = \int_1^\infty e^{\pm i\theta(x\pm i0)}\theta^{-k-1/2}d\theta = (\mp ix+0)^{k-1/2}\Gamma\left(\frac{1}{2}-k, \mp ix+0\right).$$

Choosing the branch with the positive real part for $(x\pm i0)^{1/2}$ one gets

$$(\mp ix+0)^{k-1/2} = e^{\mp i\pi(2k-1)/4}(x\pm i0)^{k-1/2},$$

and

$$\Gamma\left(\frac{1}{2}-k, \mp ix+0\right) = \Gamma\left(\frac{1}{2}-k\right) - \sum_{j=0}^\infty \frac{(-1)^j}{j!}\frac{(\mp ix+0)^{\frac{1}{2}-k+j}}{\frac{1}{2}-k+j}.$$

Thus

$$\int_1^\infty e^{\pm i\theta x}\theta^{-k-1/2}d\theta = \Gamma\left(\frac{1}{2}-k\right)e^{\mp i\pi(2k-1)/4}(x\pm i0)^{k-1/2} - \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\frac{(\mp ix+0)^j}{1/2-k+j},$$

and

$$\int_1^\infty e^{\pm i\theta x}\theta^{-k-1/2}d\theta = \Gamma\left(\frac{1}{2}-k\right)e^{\mp i\pi(2k-1)/4}(x\pm i0)^{k-1/2} \pmod{C^\infty(\mathbf{R})}.$$

Now the conclusion of the lemma is a consequence of following formula (see [3]):

$$(x\pm i0)^{k-1/2} = x_+^{k-1/2} + e^{\pm i\pi(k-1/2)}x_-^{k-1/2}.$$

□

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