Simpson's–1/3 Rule Example

Use Simpson's–1/3 rule to approximate $\displaystyle\int_0^{\pi} \sin x\, dx$ using

1. $n = 6$ subintervals,
2. $n = 12$ subintervals, and
3. Richardson extrapolation.

Here $a = 0$, $b = \pi$, and $f(x) = \sin x$.

1. $n = 6 \implies h = \dfrac{b-a}{n} = \dfrac{\pi}{6}$, and $x_i = a + ih = 0 + \frac{\pi}{6} i$, $i = 0, 1, \ldots, 6$.

i	0	1	2	3	4	5	6
x_i	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	$\frac{6\pi}{6}$
f_i	$\sin 0$	$\sin\left(\frac{\pi}{6}\right)$	$\sin\left(\frac{2\pi}{6}\right)$	$\sin\left(\frac{3\pi}{6}\right)$	$\sin\left(\frac{4\pi}{6}\right)$	$\sin\left(\frac{5\pi}{6}\right)$	$\sin\left(\frac{6\pi}{6}\right)$

So

$$\int_0^{\pi} \sin x\, dx \approx \frac{h}{3}\Big[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + 4f_5 + f_6\Big]$$
$$= 2.0008\ 6318\ 9673\ 5362\ 7840\,, \qquad \text{Error} = 4.31594 \text{ permyriad} \qquad (1)$$

(53 times smaller than Trap with $n = 6$)

2. $n = 12 \implies h = \dfrac{b-a}{n} = \dfrac{\pi}{12}$, and $x_i = a + ih = 0 + \frac{\pi}{12} i$, $i = 0, 1, \ldots, 12$.

i	0	1	2	3	$\cdots$	11	12
x_i	0	$\frac{\pi}{12}$	$\frac{2\pi}{12}$	$\frac{3\pi}{12}$	$\cdots$	$\frac{11\pi}{12}$	$\frac{12\pi}{12}$
f_i	$\sin 0$	$\sin\left(\frac{\pi}{12}\right)$	$\sin\left(\frac{2\pi}{12}\right)$	$\sin\left(\frac{3\pi}{12}\right)$	$\cdots$	$\sin\left(\frac{11\pi}{12}\right)$	$\sin\left(\frac{12\pi}{12}\right)$

So

$$\int_0^{\pi} \sin x\, dx \approx \frac{h}{3}\Big[f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 2f_{10} + 4f_{11} + f_{12}\Big]$$
$$= 2.0000\ 5262\ 4341\ 1856\ 4829\,, \qquad \text{Error} = 0.26312 \text{ permyriad} \qquad (2)$$

(217 times smaller than Trap with $n = 12$)

3. Richardson extrapolation:

$$\begin{aligned}
\text{Imp. Est.} &= \text{Better} + \frac{\text{Better} - \text{Poorer}}{2^4 - 1} \\
&= \text{Result}(2) + \frac{\text{Result}(2) - \text{Result}(1)}{2^4 - 1} \\
&= 2.0000\ 5262\ 4341\ 1856\ 4829 + (-0.0000\ 5403\ 7688\ 8233\ 7534) \\
&= 1.9999\ 9858\ 6652\ 3622\ 7295\,, \qquad \text{Error} = 0.00706 \text{ permyriad} \qquad (3)
\end{aligned}$$

NOTE: $\dfrac{\text{Error in Result}(1)}{\text{Error in Result}(2)} = \dfrac{4.31594}{0.26312} = 16.402 \approx$ is **about** 16 as expected