POLYNOMIALS CONTAINING SPECIFIED POINTS

Brute Force Algebra ("Naive Approach")

You can't appreciate the beauty of the Newton-Gregory method for constructing a polynomial through points without seeing what must be done **if we use brute force algebra**, as we would if we did not now know better. So consider the following data.

i	x_i	f_i
0	-2	40
1	1	10
2	4	-164
3	7	166

1. Suppose we construct the polynomial containing points indexed $i \in [0, 1]$. That polynomial would look like

$$P(x) \;=\; bx + a.$$

To determine the coefficients a and b using brute force algebra, we would have to **solve** these **two** equations for the **two** unknowns a and b:

$$\begin{aligned} P(-2) &= -2b + a \;=\; 40, \\ P(1) &= b + a \;=\; 10, \end{aligned}$$

to obtain $a = 20$ and $b = -10$. So the polynomial containing points indexed $i \in [0, 1]$ is

$$\boxed{P_{[0-1]}(x) \;=\; -10x + 20.} \tag{1}$$

2. Now suppose we want the polynomial containing points indexed $i \in [0, 1, 2]$. That polynomial would look like

$$P(x) \;=\; cx^2 + bx + a.$$

To determine the coefficients a, b, and c using brute force algebra, we would have to **solve** these **three** equations for the **three** unknowns a, b, and c:

$$\begin{aligned} P(-2) &= 4c - 2b + a \;=\; 40, \\ P(1) &= c + b + a \;=\; 10, \\ P(4) &= 16c + 4b + a \;=\; -164, \end{aligned}$$

to obtain $a = 36$, $b = -18$, and $c = -8$. So the polynomial containing points indexed $i \in [0, 1, 2]$ is

$$\boxed{P_{[0-2]}(x) \;=\; -8x^2 - 18x + 36.} \tag{2}$$

Notice that polynomials (1) and (2) have **no common coefficients**. That is, we cannot use polynomial $P_{[0-1]}(x)$ to build polynomial $P_{[0-2]}(x)$ as we can when using the Newton-Gregory method!

3. Now suppose we want the polynomial containing points indexed $i \in [0, 1, 2, 3]$. That polynomial would look like

$$P(x) \;=\; dx^3 + cx^2 + bx + a.$$

To determine the coefficients a, b, c, and d using brute force algebra, we would have to **solve** these **four** equations for the **four** unknowns a, b, c, and d:

$$\begin{aligned} P(-2) &= -8d + 4c - 2b + a \;=\; 40, \\ P(1) &= d + c + b + a \;=\; 10, \\ P(4) &= 64d + 16c + 4b + a \;=\; -164, \\ P(7) &= 343d + 49c + 7b + a \;=\; 166. \end{aligned}$$

to obtain $a = 68$, $b = -42$, $c = -20$ and $d = 4$. So the polynomial containing points indexed $i \in [0, 1, 2, 3]$ is

$$\boxed{P_{[0-3]}(x) \;=\; 4x^3 - 20x^2 - 42x + 68.} \tag{3}$$

Notice that polynomials (2) and (3) have **no common coefficients**. That is, we cannot use polynomial $P_{[0-2]}(x)$ to build polynomial $P_{[0-3]}(x)$ as we can when using the Newton-Gregory method!